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Morita equivalence for Banach algebras

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Abstract

We develop a theory for Morita equivalence of Banach algebras with bounded approximate identities and categories of essential modules, using functors compatible with the topology. Many aspects of discrete theory are carried over. Most importantly, the Eilenberg–Watts theorem holds, so that equivalence functors are representable as tensor functors. This enables us to determine how Banach algebras which are Morita equivalent to a given Banach algebra are constructed. It also leads to Morita invariance of bounded Hochschild homology, thus providing an efficient tool for computation of homology groups.

0. Introduction

It is a prevalent point of view to understand an algebra (or more generally, an algebraic structure) through the spaces on which it acts, that is, through its modules. This point of view dates back at least to Frobenius and the early days of representation theory. After the formation of category theory it was therefore natural to consider two algebras to be equivalent if they have equivalent categories of modules. In the paper [22] this was systematized and put to use, leading to the concept that has since been called Morita equivalence.

If the algebra in question also has a topology in which the algebraic operations are continuous and perhaps some other added structure, it is an obvious task to make a Morita theory for the subcategory of modules which are compatible with these added structures. Some twenty years after Morita's paper this was set forth in the context of von Neumann algebras and C^*-algebras by Rieffel in a series of papers, see the survey article [29] and the references therein. The relevant categories are the categories of normal modules and Hermitian modules, respectively. His elegant proof of the imprimitivity theorem was the motivating application, but Rieffel was able to use his theory in many different areas of operator algebra, both as a means of classification and as powerful tool in the understanding of concrete algebras such as transformation group C^*-algebras.

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Before we proceed, let us briefly sketch important steps in purely algebraic Morita theory. In this context the theory is developed only for unital algebras. Two unital algebras **R** and **S** are called Morita equivalent if their categories **R-mod** and **S-mod** of unit linked left modules are equivalent, that is, if there are covariant functors

$$F : \mathbf{R\text{-}mod} \longrightarrow \mathbf{S\text{-}mod},$$

$$G : \mathbf{S\text{-}mod} \longrightarrow \mathbf{R\text{-}mod}$$

and natural isomorphisms $\eta : GF \longrightarrow 1_{\mathbf{R\text{-}mod}}$ and $\zeta : FG \longrightarrow 1_{\mathbf{S\text{-}mod}}$. One then proceeds to show that these functors are representable (the Eilenberg–Watts theorem) which is to say that there are bimodules $P \in \mathbf{S\text{-}mod\text{-}R}$ and $Q \in \mathbf{R\text{-}mod\text{-}S}$ such that F is naturally equivalent to $M \longrightarrow P \otimes_{\mathbf{R}} M$ and G is naturally equivalent to $N \longrightarrow Q \otimes_{\mathbf{S}} N$. Such an equivalence module is seen to be a finitely generated projective generator.

From this one realizes that **R** and **S** are Morita equivalent if and only if $\mathbf{R} \cong pM_k(\mathbf{S})p$ and $\mathbf{S} \cong qM_l(\mathbf{R})q$ for appropriate $k, l \in \mathbb{N}$ and idempotents $p \in M_k(\mathbf{S})$ and $q \in M_l(\mathbf{R})$. Here $M_n(\star)$ denotes the algebra of $n \times n$ matrices with entries from $\star$.

The aim of this paper is to carry out the analogue of this in the context of Banach algebras. Hence the module categories should consist of Banach modules and the equivalence functors should be continuous in an appropriate sense. It turns out, see Corollary 4.13, that if one restricts attention to unital algebras, Morita theory in the Banach sense coincides with classical Morita theory. (This is also the case for strong Morita equivalence of C^*-algebras, see [29, p. 296].) Perhaps this is implicitly the reason that Morita theory in connection with the algebras of analysis other than C^*- and AW^*-algebras (see, for instance, [5, 21, 36]) has often been meant in the classical sense, that is, involving statements about matrix algebras. However, many of the algebras of functional analysis do not have identities and a theory including non-unital algebras is called for. In [33] such an algebraic (but aiming at algebras of analysis) theory is described. In the classical theory the unit is essential in important steps in the development. It is therefore reasonable to assume that in the functional analytic setting much can be recaptured by using the topology to approximate the unit. A theory from this point of view is naturally based on a topological version of the tensor product.

Hence we propose a theory for Banach algebras with bounded approximate units and their categories of essential left Banach modules and bounded module homomorphisms. Our theory, apart from covering a substantial class of Banach algebras, meets two essential qualifications of the classical theory. It is representable, that is, there is a version of the Eilenberg–Watts theorem, and among the Morita invariants is (bounded) Hochschild homology. Since the equivalence classes in general are bigger than those obtained just by forming matrix algebras, this is a promising tool for computing Hochschild homology.

There are two immediate obstacles for a transfer of the classical theory. Firstly, since the algebras are in general non-unital, there may be no projective objects in the categories of essential modules, and secondly, the categories of essential modules are not abelian, not even for finite-dimensional algebras, since the range of a morphism need not be closed, so that cokernels do not always exist. However, these two obstacles can be overcome at a rather elementary level by means of the open mapping theorem and Cohen's factorization theorem.

The questions that originally raised our interest in Morita equivalence dealt with problems of (bounded) Hochschild cohomology of algebras of compact operators [10]. We are grateful to G.A. Willis for having pointed to a possible connection to Morita theory. Some of the main results of [10] have the general form: "Let $\mathscr{A}$ be a Banach algebra with certain properties and let $\mathscr{B}$ be a subalgebra with certain properties. Then $\mathscr{A}$ is amenable if and only if $\mathscr{B}$ is amenable." In Section 7 we shall show that this can be viewed as instances of Morita invariants.

Our exposition is organized as follows. First we define Morita equivalence, find general properties as well as some elementary Morita invariants. Next we prove the Eilenberg–Watts theorem. With this at hand we are in the position to show that Hochschild homology is Morita invariant. In Section 7 we make some applications to algebras of compact operators. We finish with a discussion of examples of further areas of investigation.

The reader may notice the solid influence of the expositions [1, 4]. Throughout we shall assume basic knowledge of the language of category theory.

1. Preliminaries

In this section we will describe the module categories in which we work as well as fix notation and terminology. We shall follow usual conventions as, for example, in [12, 24] and only give details to some extent. The standard (sometimes tacit) assumption is that Banach algebras have bounded approximate identities. This provides a setting where algebraic results about unital rings can be given functional analytic versions. The usual algebraic assumption that modules are unit-linked is replaced by modules being essential: let $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ be Banach algebras. The category of the left Banach $\mathscr{A}$-modules and bounded module homomorphisms is denoted by $\mathscr{A}$-**mod**. For X and Y in $\mathscr{A}$-**mod** we denote the set of bounded module homomorphisms $X \longrightarrow Y$ by ${}_{\mathscr{A}}\mathbf{h}(X, Y)$. With the operator norm ${}_{\mathscr{A}}\mathbf{h}(X, Y)$ is a Banach space. For a module X in $\mathscr{A}$-**mod** the submodule $(\mathscr{A}.X)^-$ is called the *essential part* of X, and X is called *essential* provided $(\mathscr{A}.X)^- = X$. The full subcategory of essential modules is denoted ess-$\mathscr{A}$-**mod**. Analogously, we have the categories of right modules **mod**-$\mathscr{A}$, ess-**mod**-$\mathscr{A}$ and bimodules $\mathscr{A}$-**mod**-$\mathscr{B}$, ess-$\mathscr{A}$-**mod**-$\mathscr{B}$ with corresponding sets of module homomorphisms $\mathbf{h}_{\mathscr{A}}(X, Y)$ and ${}_{\mathscr{A}}\mathbf{h}_{\mathscr{B}}(X, Y)$. A bounded module homomorphisms $\varphi : X \longrightarrow Y$ will be called *monic* if $\varphi\gamma = 0 \Rightarrow \gamma = 0$, whenever γ is a module homomorphism with codomain X, that is, φ is monic if and only if it is injective. Similarly, *epic* is defined by the cancellation property on the right, that is, φ is epic if and only if it has dense range. A monic (epic) is an *embedding* (*epimorphism*) if it has closed range. The notations ${}_{\mathscr{A}}X$, $X_{\mathscr{A}}$ and ${}_{\mathscr{A}}X_{\mathscr{B}}$ are shorthand indications that X is in $\mathscr{A}$-**mod**, **mod**-$\mathscr{A}$ and

$\mathscr{A}$**-mod-**$\mathscr{B}$, respectively. For X in **mod-**$\mathscr{A}$ and Y in $\mathscr{A}$**-mod** we define the topological tensor product $X \hat{\otimes}_{\mathscr{A}} Y$ as in [24]. It is the universal object for $\mathscr{A}$-balanced bounded bilinear maps from $X \times Y$ and can be realized as $X \hat{\otimes} Y / U$ where $\hat{\otimes}$ is the projective tensor product and $U = \operatorname{clspan}\{x.a \otimes y - x \otimes a.y \mid x \in X,\ y \in Y,\ a \in \mathscr{A}\}$. For ${}_{\mathscr{B}}X_{\mathscr{A}}$ and ${}_{\mathscr{A}}Y_{\mathscr{C}}$ the tensor product $X \hat{\otimes}_{\mathscr{A}} Y$ is naturally in $\mathscr{B}$**-mod-**$\mathscr{C}$ and "being essential" follows suit. The canonical map $\pi_Y : \mathscr{A} \hat{\otimes}_{\mathscr{A}} Y \longrightarrow Y$ given by $a \otimes_{\mathscr{A}} y \longrightarrow a.y$ is of vital importance. We have the following proposition.

Proposition 1.1 (Rieffel [24]). *If $\mathscr{A}$ has a bounded left approximate identity and $Y \in$ ess-$\mathscr{A}$**-mod**, then π_Y is an isomorphism of modules.*

Proof. See [24]. □

The object of study in this paper is categories of essential modules over Banach algebras with bounded approximate identities. As pointed out in the introduction, since algebras of interest are not necessarily unital, and hence not free, left modules, projective objects may fail to exist. However, from the existence of a bounded approximate identity we may retrieve enough of the projectivity property for our purpose. First we make a definition.

Definition 1.2. Let $P \in \mathscr{A}$**-mod**. We call P *approximately projective* if to every test diagram in $\mathscr{A}$**-mod**

$$\begin{array}{ccccc} & & P & & \\ & & \Big\downarrow{\scriptstyle \phi} & & \\ Y & \xrightarrow{q} & Z & \longrightarrow & 0 \end{array}$$

where $q : Y \longrightarrow Z$ is an epimorphism, there is a bounded net of module homomorphisms $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda}$, $\tilde{\varphi}_\lambda : P \longrightarrow Y$ such that $q\tilde{\varphi}_\lambda \longrightarrow \varphi$ in the bounded strong operator topology (b.s.o.).

Proposition 1.3. *If $\mathscr{A}$ has a bounded right approximate identity, then ${}_{\mathscr{A}}\mathscr{A}$ is approximately projective.*

Proof. Let $(e_\lambda)_{\lambda \in \Lambda}$ be a bounded right approximate identity for $\mathscr{A}$ and let

$$\begin{array}{ccccc} & & \mathscr{A} & & \\ & & \Big\downarrow{\scriptstyle \phi} & & \\ Y & \xrightarrow{q} & Z & \longrightarrow & 0 \end{array}$$

be a test diagram in $\mathscr{A}$**-mod**. By the open mapping theorem there is a bounded net $(y_\lambda)_{\lambda \in \Lambda}$ in Y such that $q(y_\lambda) = \varphi(e_\lambda)$ $(\lambda \in \Lambda)$. Define $\tilde{\varphi}_\lambda : \mathscr{A} \longrightarrow Y$ by $\tilde{\varphi}_\lambda(a) = a.y_\lambda$.

Then $q(\tilde{\varphi}_\lambda(a)) = \varphi(ae_\lambda)$, so that $q\tilde{\varphi}_\lambda \longrightarrow \tilde{\varphi}$ (b.s.o.). Since obviously $\tilde{\varphi}_\lambda \in {}_{\mathscr{A}}\mathbf{h}(\mathscr{A}, Y)$, it follows that $\mathscr{A}$ is approximately projective. □

Let us recall the definitions of the homological concepts flat and injective. A module $P \in \mathscr{A}$**-mod** is called *flat* if the functor $- \hat{\otimes}_{\mathscr{A}} P$ is exact, that is, if

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

is any short exact sequence in **mod-**$\mathscr{A}$, then

$$0 \longrightarrow X \hat{\otimes}_{\mathscr{A}} P \longrightarrow Y \hat{\otimes}_{\mathscr{A}} P \longrightarrow Z \hat{\otimes}_{\mathscr{A}} P \longrightarrow 0$$

is a short exact sequence of Banach spaces. Similarly, a module $Q \in \mathscr{A}$**-mod** is called *injective*, if the functor ${}_{\mathscr{A}}\mathbf{h}(-, Q)$ is exact. Analogously, we define flatness and injectivity of right Banach modules.

For $P \in \mathscr{A}$**-mod** the space P^* of bounded linear functionals is naturally in **mod-**$\mathscr{A}$. The dual actions are defined by

$$\langle x, x^*.a \rangle = \langle a.x, x^* \rangle \quad (a \in \mathscr{A},\ x \in P,\ x^* \in P^*).$$

It is standard that ${}_{\mathscr{A}}P$ is flat, if and only if $P^*_{\mathscr{A}}$ is injective and that ${}_{\mathscr{A}}Q$ is injective, if and only if every diagram in $\mathscr{A}$**-mod**

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"'] \arrow[d, "\psi"'] & Y \\ & Q & \end{tikzcd}$$

with $i(X)$ closed can be completed by $\tilde{\psi} \in {}_{\mathscr{A}}\mathbf{h}(Y, Q)$ to make it commutative. Analogous statements hold for right Banach modules.

Proposition 1.4. *Let* $P \in$ ess-$\mathscr{A}$**-mod** *be approximately projective. Then* P *is flat.*

Proof. We prove that $P^* \in$ **mod-**$\mathscr{A}$ is injective. (Note that P^* is not necessarily essential.) Let

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"'] \arrow[d, "\psi"'] & Y \\ & P^* & \end{tikzcd}$$

be a test diagram in **mod-**$\mathscr{A}$ with $i(X)$ closed. Then we have

$$\begin{tikzcd} & P \arrow[d] & \\ & P^{**} \arrow[d, "\psi^*"] & \\ Y^* \arrow[r] & X^* \arrow[r] & 0 \end{tikzcd}$$

where $P \longrightarrow P^{**}$ is the canonical embedding. Let $(\tilde{\varphi}_\lambda)_{\lambda \in \Lambda}$ be a net in ${}_{\mathscr{A}}\mathbf{h}(P, Y^*)$ completing the diagram and let $\tilde{\varphi}: P \longrightarrow Y^*$ be a bounded weak operator topology (b.w.o.) accumulation point (identifying $\mathscr{B}(P, Y^*)$ with $(P \hat{\otimes} Y)^*$). Let $j: Y \longrightarrow Y^{**}$ be the canonical embedding. Then $\tilde{\psi} = \tilde{\varphi}^* \circ j$ completes the first diagram. □

Remark. The homological concepts projective, flat, injective, etc., are defined by means of the usual test diagrams. We do not consider "admissable" test diagrams as in the homology theory of Helemskii [12] since "admissability" is, except for trivial cases, not a functorial property. Our concepts are, in the terminology of Helemskii, strictly projective, strictly flat, etc.

Definition 1.5. Let $\mathbf{C}$ be a subcategory of $\mathscr{A}$**-mod** and let $(M_\gamma)_{\gamma \in \Gamma}$ be a family of modules in $\mathbf{C}$. A *direct sum* of (M_γ) is a module M in $\mathbf{C}$ and a family of homomorphisms in $\mathbf{C}$: $(i_\gamma)_{\lambda \in \Gamma}$, $i_\gamma: M_\gamma \longrightarrow M$ with $\sup_{\gamma \in \Gamma} \| i_\gamma \| < \infty$ such that for all modules X in $\mathbf{C}$ and all families of homomorphisms $\varphi_\gamma: M_\gamma \longrightarrow X$ $(\gamma \in \Gamma)$ in $\mathbf{C}$ with $\sup_{\gamma \in \Gamma} \| \varphi_\gamma \| < \infty$ there is a unique homomorphism $\varphi: M \longrightarrow X$ in $\mathbf{C}$ such that $\varphi \circ i_\gamma = \varphi_\gamma$ $(\gamma \in \Gamma)$. The dual concept *direct product* is defined by reversing arrows. We use the notation $\bigoplus_{\gamma \in \Gamma} M_\gamma$ for direct sum and $\prod_{\gamma \in \Gamma} M_\gamma$ for direct product suppressing the "coordinate-maps" (i_γ) as is customary.

Standard arguments give that direct sums and products, if they exist, are unique up to isomorphism.

Proposition 1.6. *The category* ess-$\mathscr{A}$**-mod** *has direct sums and products.*

Proof. The direct sum $\bigoplus_{\gamma \in \Gamma} M_\gamma$ can be realized as the ℓ_1-sum of the modules M_γ with coordinatewise module operations. It is clear that we obtain an essential module in this way. To define a direct product first form the cartesian product $\times_{\gamma \in \Gamma} M_\gamma$ and make it an object in $\mathscr{A}$**-mod** by means of the coordinatewise operations and the ℓ_∞-norm. Define $\prod_{\gamma \in \Gamma} M_\gamma$ to be the essential part $(\mathscr{A} . \times_{\gamma \in \Gamma} M_\gamma)^-$. □

In the next definition we summarize some important module concepts.

Definition 1.7. Let $\mathbf{C}$ be a subcategory of $\mathscr{A}$**-mod** and let M be a module in $\mathbf{C}$.

(1) M is called *faithful* if $a.M = \{0\} \Rightarrow a = 0$.

(2) M *generates* $X \in \mathbf{C}$, if there is a direct sum $\bigoplus_{\gamma \in \Gamma} M$ of copies of M and an epimorphism $\bigoplus_{\gamma \in \Gamma} M \xrightarrow{\varphi} X$ in $\mathbf{C}$. M is a *generator* for $\mathbf{C}$, if M generates all modules $X \in \mathbf{C}$.

(3) M *cogenerates* $X \in \mathbf{C}$ if there is a direct product $\prod_{\gamma \in \Gamma} M$ of copies of M and an embedding $0 \longrightarrow X \longrightarrow \prod_{\gamma \in \Gamma} M$ in $\mathbf{C}$. M is a *cogenerator* for $\mathbf{C}$ if it cogenerates all X in $\mathbf{C}$.

Proposition 1.8. *If $\mathscr{A}$ has a bounded left approximate identity of bound d, then ${}_{\mathscr{A}}\mathscr{A}$ is a generator for* ess-$\mathscr{A}$**-mod**. *More precisely, for each $X \in$* ess-$\mathscr{A}$**-mod** *there is an epimorphism $\bigoplus_{\gamma \in \Gamma} \mathscr{A} \xrightarrow{\varphi} X$ with inversion constant not exceeding d.*

Proof. Let Γ be the unit ball of X and let $\varphi : \bigoplus_{\gamma \in \Gamma} \mathscr{A} \longrightarrow X$ be the lift of the maps $\varphi_\gamma(a) = a.\gamma$. By Cohen's factorization theorem φ is an epimorphism. More precisely, if $\|x\| < 1$, then there is $\gamma \in \Gamma$ and $a \in \mathscr{A}$ with $\|a\| \le d$ and $x = a\gamma$ [3, p. 61]. $\square$

Proposition 1.9. *Suppose that $\mathscr{A}$ has a bounded approximate identity. Then $U \in$* ess-$\mathscr{A}$-**mod** *is faithful if and only if it cogenerates a generator.*

Proof. Suppose U is faithful. Let Γ be the unit ball in U. Then the map $\mathscr{A} \longrightarrow \prod_{\gamma \in \Gamma} U : a \longrightarrow (a.\gamma)_{\gamma \in \Gamma}$ is monic. Since $\sup \|a.\gamma\| \ge (1/d)\|a\|$, d being the bound of the approximate identity, it has closed range, so U cogenerates the generator $\mathscr{A}$. If U cogenerates M then $a.U = \{0\} \Rightarrow a.M = \{0\}$. If M generates $\mathscr{A}$, we get $a.U = \{0\} \Rightarrow a.\mathscr{A} = \{0\}$. But clearly, since $\mathscr{A}$ has a bounded approximate identity, $a.\mathscr{A} = \{0\} \Leftrightarrow a = 0$. $\square$

The endomorphism algebra $\mathbf{h}_{\mathscr{A}}(\mathscr{A}, \mathscr{A})$ will be denoted $\mathscr{L}(\mathscr{A})$, the *left centralizer algebra*, and the endomorphism algebra ${}_{\mathscr{A}}\mathbf{h}(\mathscr{A}, \mathscr{A})$ will be denoted $\mathscr{R}(\mathscr{A})$, the *right centralizer algebra*. The *double centralizer algebra* is $\mathscr{M}(\mathscr{A}) = \{(S, T) \in \mathscr{L}(\mathscr{A}) \times \mathscr{R}(\mathscr{A}) \mid aS(b) = T(a)b,\ a, b \in \mathscr{A}\}$. The maps $a \longrightarrow L_a$, ($a \longrightarrow R_a$, $a \longrightarrow (L_a, R_a)$) embed $\mathscr{A}$ as a right (left, two-sided) closed ideal.

Proposition 1.10. *Let $M \in$* ess-$\mathscr{A}$**-mod**. *Then the module action can be extended to $\mathscr{L}(\mathscr{A})$. Similarly for right modules and two-sided modules.*

Proof. Let $T \in \mathscr{L}(\mathscr{A})$. For $m \in M$ define $T.m = \lim_\lambda Te_\lambda.m$ where $(e_\lambda)_{\lambda \in \Lambda}$ is a bounded (left-) approximate identity for $\mathscr{A}$, see [14]. $\square$

2. Definition and basic properties of equivalence

Throughout, $\mathscr{A}$ and $\mathscr{B}$ will denote Banach algebras with bounded two-sided approximate identities and ess-$\mathscr{A}$**-mod**, ess-$\mathscr{B}$**-mod** will denote their respective categories of essential Banach left modules. Functors between two such categories will always be assumed to be additive. Since we would also like to respect functional analytic properties we make the following definition.

Definition 2.1. Let $F : \text{ess-}\mathscr{A}\textbf{-mod} \longrightarrow \text{ess-}\mathscr{B}\textbf{-mod}$ be a (covariant) functor. We say that F is *bounded* if

$$\sup\{\|F_{M,N}\| \mid M, N \in \text{ess-}\mathscr{A}\textbf{-mod}\} < \infty ,$$

where $F_{M,N} : {}_{\mathscr{A}}\mathbf{h}(M,N) \longrightarrow {}_{\mathscr{B}}\mathbf{h}(FM,FN)$ is F applied to homomorphisms, and we say that F is *strongly continuous* if $F_{M,N}$ is continuous with respect to the strong operator topologies on ${}_{\mathscr{A}}\mathbf{h}(M,N)$ and ${}_{\mathscr{B}}\mathbf{h}(FM,FN)$ for all choices of $M, N \in \text{ess-}\mathscr{A}\textbf{-mod}$.

Likewise, to compare bounded functors we need natural transformations to be bounded.

Definition 2.2. Let $\eta : F \longrightarrow G$ be a natural transformation between two (bounded) functors $F, G : \text{ess-}\mathscr{A}\textbf{-mod} \longrightarrow \text{ess-}\mathscr{B}\textbf{-mod}$. Then η is *bounded* provided

$$\sup\{\|\eta_M\| \mid M \in \text{ess-}\mathscr{A}\textbf{-mod}\} < \infty ,$$

where $\eta_M : FM \longrightarrow GM$. If all η_M's are bijective we say that η is a *bounded natural isomorphism* provided η is bounded and

$$\sup\{\|\eta_M^{-1}\| \mid M \in \text{ess-}\mathscr{A}\textbf{-mod}\} < \infty .$$

Equipped with these definitions we can now state the following definition.

Definition 2.3. Two Banach algebras $\mathscr{A}$ and $\mathscr{B}$ are *Morita equivalent* if there are bounded strongly continuous functors

$$\text{ess-}\mathscr{A}\textbf{-mod} \underset{G}{\overset{F}{\rightleftarrows}} \text{ess-}\mathscr{B}\textbf{-mod}$$

and bounded natural isomorphisms

$$\eta : GF \longrightarrow \mathbf{1}_{\text{ess-}\mathscr{A}\text{-mod}}, \qquad \zeta : FG \longrightarrow \mathbf{1}_{\text{ess-}\mathscr{B}\text{-mod}} .$$

Such F and G will be called *equivalence functors*. If $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent we write $\mathscr{A} \approx \mathscr{B}$.

The fundamental properties on which most of the theory rests, are that an equivalence functor is an isomorphism on hom-spaces and that, if F and G are respective inverse equivalence functors, then both (F, G) and (G, F) are adjoint pairs of functors. The Banach algebra version of these statements is described in the following two propositions. The proofs follow closely the corresponding algebraic proofs, see, for instance, [1], so we give only sketches.

Proposition 2.4. *Let $F : \text{ess-}\mathscr{A}\textbf{-mod} \longrightarrow \text{ess-}\mathscr{B}\textbf{-mod}$ be an equivalence functor. Then the restrictions to hom-spaces*

$$F_{M,N} : {}_{\mathscr{A}}\mathbf{h}(M,N) \longrightarrow {}_{\mathscr{B}}\mathbf{h}(FM,FN)$$

are isomorphisms of Banach spaces and isomorphisms with respect to b.s.o. topologies such that Ff is monic (epic) if and only if f is monic (epic). Furthermore, $\sup\{\|F_{MN}^{-1}\| \mid M,N \in \text{ess-}\mathscr{A}\textbf{-mod}\} < \infty$.

Proof. The inverses $H : {}_{\mathscr{B}}\mathbf{h}(FM,FN) \longrightarrow {}_{\mathscr{A}}\mathbf{h}(M,N)$ are given by

$$Hf = \eta_N G f \eta_M^{-1},$$

where G is the inverse equivalence functor. Monic and epic are purely categorial terms. □

Proposition 2.5. *Let $\mathscr{A}$ and $\mathscr{B}$ be Morita equivalent with F and G as above. Then there are bounded natural isomorphisms*

$$\phi : {}_{\mathscr{B}}\mathbf{h}(N,FM) \longrightarrow {}_{\mathscr{A}}\mathbf{h}(GN,M),$$

$$\theta : {}_{\mathscr{B}}\mathbf{h}(FM,N) \longrightarrow {}_{\mathscr{A}}\mathbf{h}(M,GN)$$

(bounded and natural in each variable). These are also isomorphisms with respect to b.s.o. topologies. Furthermore, $\phi(f)$ is monic (epic) if and only if f is monic (epic), and likewise for θ.

Proof. The bounded natural isomorphism ϕ is given by

$$\phi(f) = \eta_M G f \quad (f \in {}_{\mathscr{B}}\mathbf{h}(N,FM))$$

and θ is given by

$$\theta(f) = G f \eta_M^{-1}.$$

That, ϕ say, is natural in both variables means that $\phi(Fh\gamma k) = h\phi(\gamma)Gk$, whenever the compositions make sense. The boundedness and preservation of monics and epics are immediate from Proposition 2.4. □

Many constructions preserved by Morita equivalence can be viewed as instances of universal elements. The two propositions above are powerful tools, in particular because we can use them to show that certain universal elements are preserved by Morita equivalence. To be specific we make an appropriate definition, (see also [20]). Let **Set** be the category of sets and recall that if $H : \mathbf{C} \longrightarrow \mathbf{Set}$ is a (co-/contravariant) functor from a category $\mathbf{C}$, then a universal element for H is a pair $\langle e, U \rangle$ where U is an object in $\mathbf{C}$ and $e \in HU$ such that for all pairs $\langle f, W \rangle$, where W is an object in $\mathbf{C}$ and $f \in HW$, there is a unique morphism γ in $\mathbf{C}$ such that $H(\gamma)e = f$. We shall be concerned with functors within the category of modules of the following type.

Definition 2.6. Let $(M_i)_{i\in I}$ be a family of modules in ess-$\mathscr{A}$-**mod**. We say that a (covariant) functor is *associated* to the family $(M_i)_{i\in I}$ if it is defined on the full category ess-$\mathscr{A}$-**mod** and for each $M \in$ ess-$\mathscr{A}$-**mod**

$$H(M) \subseteq \bigtimes_{i\in I} ({}_{\mathscr{A}}\mathbf{h}(M_i, M))$$

and for each homomorphism γ,

$$H\gamma(f_i)_{i\in I} = (\gamma f_i)_{i\in I}.$$

The contravariant case is defined analogously. Here $\times_{i\in I}$ stands for cartesian product.

Now suppose F: ess-$\mathscr{A}$-**mod** $\longrightarrow$ ess-$\mathscr{B}$-**mod** has a right adjoint G with bounded natural isomorphism $\theta: {}_{\mathscr{B}}\mathbf{h}(F((\cdot),\cdot) \longrightarrow {}_{\mathscr{A}}\mathbf{h}(\cdot, G(\cdot))$ and that H is associated to $(M_i)_{i\in I}$. Then we define the covariant functor H_F with domain ess-$\mathscr{B}$-**mod** by

$$H_F(N) = \left\{ (f_i)_{i\in I} \in \bigtimes_{i\in I} ({}_{\mathscr{B}}\mathbf{h}(F(M_i), N)) \,|\, (\theta f_i)_{i\in I} \in H(GN) \right\},$$

and

$$[H_F(\gamma)](f_i)_{i\in I} = (\gamma f_i)_{i\in I}.$$

The contravariant case is defined analogously.

Theorem 2.7. *Let H be associated to a family $(M_i)_{i\in I}$* and let F: ess-$\mathscr{A}$-**mod** $\longrightarrow$ ess-$\mathscr{B}$-**mod** *be an equivalence. Then H_F is associated to $(FM_i)_{i\in I}$, and $\langle (e_i)_{i\in I}, U\rangle$ is a universal element for H if and only if $\langle (Fe_i)_{i\in I}, FU\rangle$ is a universal element for H_F.*

Proof. The only non-trivial part of the first statement is that H_F is defined on the full category ess-$\mathscr{B}$-**mod**. To see this let $\gamma \in {}_{\mathscr{B}}\mathbf{h}(N_1, N_2)$ and let $(f_i)_{i\in I} \in H_F(N_1)$. Then $(\theta f_i)_{i\in I} \in H(GN_1)$ and therefore $(\theta(\gamma f_i))_{i\in I} = (G(\gamma)\theta f_i)_{i\in I} = H(G(\gamma))((\theta f_i)_{i\in I}) \in H(GN_2)$. Hence H_F is associated to $((FM_i)_{i\in I}$. Now let $N \in$ ess-$\mathscr{B}$-**mod** and let $(f_i)_{i\in I} \in H_F(N)$. Since a factorization

$$\begin{tikzcd} FM_i \arrow[rr, "Fe_i"] \arrow[dr, "f_i"'] & & FU \arrow[dl, "b"] \\ & N & \end{tikzcd} \qquad (i \in I)$$

exists, if and only if the factorization

$$\begin{tikzcd} M_i \arrow[rr, "e_i"] \arrow[dr, "\theta f_i"'] & & U \arrow[dl, "a"] \\ & G(N) & \end{tikzcd} \qquad (i \in I)$$

with $\theta(b) = a$ exists, it follows that if $\langle (e_i)_{i \in I}, U \rangle$ is universal for H, then $\langle (Fe_i)_{i \in I}, FU \rangle$ is universal for H_F. Suppose now that $\langle (Fe_i)_{i \in I}, FU \rangle$ is universal for H_F and let $M \in$ ess-$\mathscr{A}$-**mod** and $(g_i)_{i \in I} \in HM$. Then we have, applying θ^{-1}, a unique factorization $\theta^{-1}(\eta^{-1} g_i) = bFe_i$ $(i \in I)$. Hence $g_i = \eta\theta(b)e_i$ $(i \in I)$ and again uniqueness follows as above. $\square$

Corollary 2.8. *Let F : ess-$\mathscr{A}$-**mod** $\longrightarrow$ ess-$\mathscr{B}$-**mod** be an equivalence and let (M_k, φ_{ij}) be a direct system.*

Then $(FM_k, F\varphi_{ij})$ is a direct system and

$$F\left(\varprojlim M_k\right) = \varprojlim F(M_k),$$

$$F\left(\varinjlim M_k\right) = \varinjlim F(M_k).$$

In particular, F preserves direct sums and products.

Proof. $\varinjlim M_k$ is the universal element for the functor

$$H(M) = \{(f_k)_{k \in K} \mid f_k \in {}_{\mathscr{A}}\mathbf{h}(M_k, M), \sup \| f_k \| < \infty, f_i f_j \varphi_{ij}\}.$$

Since F is bounded it follows that $\varinjlim FK_k$ is the universal element for the functor H_F. The inverse limit statement follows analogously. $\square$

We shall need a little more detail on direct sums. As is well known these can be specified to within isometric isomorphism, when the universal property is defined by diagrams

$$\begin{tikzcd} \bigoplus M_j \arrow[rr, "f"] & & M \\ & M_k \arrow[ul, "\iota_k"] \arrow[ur, "f_k"'] & \end{tikzcd}$$

requiring $\| f \| = \sup\{ \| f_k \| \}$. This is obtained by considering the functor

$$H(M) = \{(f_k) \mid f_k \in {}_{\mathscr{A}}\mathbf{h}(M_k, M), \| f_k \| \leq 1\},$$

defined on the (non-additive) category of essential modules and only contractions as morphisms.

The obstacle of the category $\mathscr{A}$-**mod** not being abelian, mentioned in the introduction, manifests itself in the difference between being epic and epimorphic. Whereas only general category theory arguments were needed to show that equivalences preserve the property of being epic, extra arguments are needed to deal with epimorphisms. We are now in the position to do this.

Corollary 2.9. *Let*

$$\mathscr{C}: 0 \longrightarrow X \xrightarrow{\ i\ } Y \xrightarrow{\ q\ } Z \longrightarrow 0$$

be a short complex in ess-$\mathscr{A}$-**mod**. *Then* $\mathscr{C}$ *is (split-) exact if and only if* $F(\mathscr{C})$ *is (split-) exact.*

Proof. The split-part follows from F being additive. The corollary now follows from viewing short exactness as a universal property: The complex $\mathscr{C}$ is exact if and only if $\langle i, X \rangle$ is the universal element for the functor

$$H_1(M) = \{ f \in {}_{\mathscr{A}}\mathbf{h}(M, Y) \mid q \circ f = 0 \}$$

and $\langle q, Z \rangle$ is the universal element for the functor

$$H_2(M) = \{ f \in {}_{\mathscr{A}}\mathbf{h}(Y, M) \mid f \circ i = 0 \}.$$

To see this, first suppose that $\langle i, X \rangle$ and $\langle q, Z \rangle$ are universal elements as described. From the factorizations of the canonical map $X/\ker i \longrightarrow Y$ and the inclusion $\ker q \longrightarrow Y$, we see that i is monic and $\operatorname{im} i = \ker q$. Now if $\operatorname{im} i = \ker q$, then $\langle Q, Y/\ker q \rangle$, $Q: Y \longrightarrow Y/\ker q$ being the natural map, is the universal element for H_2. Hence there is an isomorphism $\tilde{q}: Y/\ker q \longrightarrow Z$, such that $q = \tilde{q} \circ Q$. In particular q is surjective. Conversely, suppose $\mathscr{C}$ is short exact and put $K = \operatorname{im} i = \ker q$. By the open mapping theorem there are isomorphisms $s \in {}_{\mathscr{A}}\mathbf{h}(K, X)$ and $t \in {}_{\mathscr{A}}\mathbf{h}(Z, Y/K)$ such that $si = \mathrm{id}_X$ and $Q = tq$. It follows that $\langle i, X \rangle$ is universal for H_1 and $\langle q, Z \rangle$ is universal for H_2. $\square$

Corollary 2.10. *Let F be an equivalence functor and let ϕ and θ be as in Proposition* 2.5. *Then $F\gamma$, $\phi(f)$ and $\theta(g)$ are embeddings (epimorphisms) if and only if γ, f and g are embeddings (epimorphisms).*

3. The Eilenberg–Watts theorem

In pure algebra the Eilenberg–Watts theorem [9, 34] states that a right exact direct sum preserving functor between module categories is, up to natural equivalence, given by tensoring. This is fairly straightforward, using that the tensor functor is right exact and preserves direct sums. It is easy to see that in the Banach algebra case a tensor functor also preserves direct sums. However, since ess-$\mathscr{A}$-**mod** is not an abelian category this functor is not in general right exact. Still, the algebraic proof of exactness can be adapted to give the following lemma.

Lemma 3.1. *Let* $0 \longrightarrow X \xrightarrow{i} Y \xrightarrow{q} Z \longrightarrow 0$ *be a short exact sequence in* $\mathscr{A}$-**mod** *and let* $P \in$ **mod**-$\mathscr{A}$. *Then*

$$0 \longrightarrow P \hat{\otimes}_{\mathscr{A}} X \longrightarrow P \hat{\otimes}_{\mathscr{A}} Y \longrightarrow P \hat{\otimes}_{\mathscr{A}} Z \longrightarrow 0$$

is exact at $P \hat{\otimes}_{\mathscr{A}} X$ *and* $\operatorname{im}(\mathbf{1} \hat{\otimes}_{\mathscr{A}} i)$ *is dense in* $\ker(\mathbf{1} \hat{\otimes}_{\mathscr{A}} q)$.

Proof. Clearly $\mathbf{1} \hat{\otimes}_{\mathscr{A}} q$ is surjective: Let $\sum \|p_i\| \, \|z_i\| < \infty$. By the open mapping theorem we may choose $y_i \in Y$ so that $q(y_i) = z_i$ and $\sum \|p_i\| \, \|y_i\| < \infty$. It follows that $(\mathbf{1} \hat{\otimes}_{\mathscr{A}} q)(\sum p_i \otimes_{\mathscr{A}} y_i) = \sum p_i \hat{\otimes}_{\mathscr{A}} z_i$. It is also clear that $\operatorname{im}(\mathbf{1} \hat{\otimes}_{\mathscr{A}} i) \subseteq \ker(\mathbf{1} \hat{\otimes}_{\mathscr{A}} q)$. Hence let $K = (\operatorname{im}(\mathbf{1} \hat{\otimes}_{\mathscr{A}} i))^{-}$ and consider the bilinear map $\beta : P \times Z \longrightarrow P \hat{\otimes}_{\mathscr{A}} Y/K$ given by $(p, z) \longrightarrow p \otimes_{\mathscr{A}} y + K$, where $q(y) = z$. Then β is well defined because, if $q(y) = 0$, then $y = i(x)$ so $p \otimes_{\mathscr{A}} y \in K$. Furthermore, it is obviously balanced, and since q is open, it is bounded. It follows that there is a surjective bounded linear map $\bar{\beta} : P \hat{\otimes}_{\mathscr{A}} Z \longrightarrow P \hat{\otimes}_{\mathscr{A}} Y/K$ such that, if $\alpha : P \hat{\otimes}_{\mathscr{A}} Y/\ker(\mathbf{1} \hat{\otimes}_{\mathscr{A}} q) \longrightarrow P \hat{\otimes}_{\mathscr{A}} Z$ is the canonical isomorphism induced by q, we get a commutative diagram

$$\begin{tikzcd}
P \hat{\otimes}_{\mathscr{A}} Y/\ker(\mathbf{1} \hat{\otimes}_{\mathscr{A}} q) \arrow[rr, "\simeq", "\alpha"'] & & P \hat{\otimes}_{\mathscr{A}} Z \arrow[dl, "\bar{\beta}"] \\
& P \hat{\otimes}_{\mathscr{A}} Y/K \arrow[ul, "Q"] &
\end{tikzcd}$$

with $\alpha Q \bar{\beta} = \mathrm{id}_{P \hat{\otimes}_{\mathscr{A}} Z}$. Hence $\bar{\beta}$ is also injective, i.e. an isomorphism. It follows that $K = \ker(\mathbf{1} \hat{\otimes}_{\mathscr{A}} q)$. $\square$

In order to proceed, we shall need the following functional analytic version of the five-lemma.

Lemma 3.2. *Suppose we have a commutative diagram of topological vector spaces and continuous linear maps*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\tau_1"'] & Y \arrow[r, "g"] \arrow[d, "\tau_2"'] & Z \arrow[r] \arrow[d, "\tau_3"'] & 0 \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r] & 0
\end{tikzcd}$$

with lower row exact, upper row exact at Z, *and* $(\operatorname{im} f)^{-} = \ker g$. *Suppose further that* τ_1 *and* τ_2 *are topological isomorphisms. Then* $\operatorname{im} f = \ker g$ *and* τ_3 *is an isomorphism, which is topological if* g' *is open.*

Proof. Since $\operatorname{im} f = \tau_2^{-1}(\operatorname{im} f')$ is actually closed, this follows from the usual five-lemma and the fact that τ_3 is open when g' is. $\square$

Theorem 3.3 (Eilenberg–Watts). *Let F: ess-$\mathscr{A}$-**mod** $\longrightarrow$ ess-$\mathscr{B}$-**mod** be an equivalence functor. Then there is ${}_{\mathscr{B}}P_{\mathscr{A}} \in$ ess-$\mathscr{B}$-**mod**-$\mathscr{A}$ such that $F \cong {}_{\mathscr{B}}P \hat{\otimes}_{\mathscr{A}} -$.*

Proof. Let ${}_{\mathscr{B}}P = F(\mathscr{A})$. Then we make P a right Banach $\mathscr{A}$-module by $p.a = F(R_a)(p)$, where $R_a : x \longrightarrow xa$ is the element in the embedding of $\mathscr{A}$ into ${}_{\mathscr{A}}\mathbf{h}(\mathscr{A},\mathscr{A})$. Since F is strongly continuous, $P_{\mathscr{A}}$ is essential. Let $\Phi = {}_{\mathscr{B}}P \hat{\otimes}_{\mathscr{A}} -$. We define a natural transformation $\tau : \Phi \longrightarrow F$ by $\tau_M : P \hat{\otimes}_{\mathscr{A}} M \longrightarrow F(M)$ where

$$\tau_M(p \otimes_{\mathscr{A}} x) = F(R_x)(p) \quad (p \in P, x \in M)$$

and $R_x : a \longrightarrow a.x \in {}_{\mathscr{A}}\mathbf{h}(\mathscr{A}, M)$. Then τ is easily seen to be natural and $\|\tau_M\| \le \|F\|$. So it is bounded. Furthermore,

$$\tau_A(p \otimes_{\mathscr{A}} a) = p.a$$

so that $\tau_A : P \hat{\otimes}_{\mathscr{A}} \mathscr{A} \longrightarrow F(\mathscr{A})$ is the Rieffel isomorphism. Next we show that τ is an isomorphism on direct sums of copies of $\mathscr{A}$. Let $S = \bigoplus \mathscr{A}_\alpha$, $\mathscr{A}_\alpha = \mathscr{A}$. Since τ is natural, we have commutative diagrams

$$\begin{array}{ccc} F(S) & \xleftarrow{\tau_S} & \Phi(S) \\ {\scriptstyle F(i_\alpha)}\uparrow & & \uparrow{\scriptstyle \Phi(i_\alpha)} \\ F(\mathscr{A}) & \xleftarrow[\tau_A]{} & \Phi(\mathscr{A}) \end{array}$$

Since both F (by Corollary 2.8) and Φ are direct sum preserving, we see that τ_S is an isomorphism with inverse being the lift of $(\tau_A)^{-1}$. Referring to the proof of Theorem 2.6 we see specifically that $\tau_S^{-1} = \theta^{-1}(a)$, where a is the lift of $\theta(\Phi(i_\alpha)\tau_A^{-1})$. If we have chosen the contraction version of direct sums as described after Corollary 2.8 we get the estimate $\|\tau_S^{-1}\| \le \|\theta\| \|\theta^{-1}\| \|\tau_A^{-1}\|$.

Now let M be arbitrary. Since $\mathscr{A}$ is a generator there is an exact sequence

$$S_1 \xrightarrow{f} S_2 \xrightarrow{g} M \longrightarrow 0,$$

with S_1 and S_2 direct sums of copies of $\mathscr{A}$, $\|f\|, \|g\| \le 1$ and such that the inversion constant of g is smaller than d, the bound of the approximate identity. Applying F and Φ we get a commutative diagram

$$\begin{array}{ccccccc} \Phi(S_1) & \longrightarrow & \Phi(S_2) & \longrightarrow & \Phi(M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \tau_{S_1}} & & \downarrow{\scriptstyle \tau_{S_2}} & & \downarrow{\scriptstyle \tau_M} & & \\ F(S_1) & \longrightarrow & F(S_2) & \longrightarrow & F(M) & \longrightarrow & 0 \end{array}$$

By Corollary 2.9 and Lemmas 3.1 and 3.2, τ_M is an isomorphism and, using the bound of the inversion constant of g, we have $\|\tau_M^{-1}\| \le d\|F\|\|\tau_{S_2}^{-1}\|\|\theta^{-1}\|$, so that $\sup_M\|\tau_M^{-1}\| < \infty$. It follows that τ is a bounded natural isomorphism. $\square$

Remark. In the definition of an equivalence functor we included the condition that it be continuous with respect to b.s.o. topologies. Since tensor functors do have this property, the condition is implicit, if we want the Eilenberg–Watts theorem to hold. On the other hand, the condition is explicitly used to obtain that the right action of $\mathscr{A}$ on ${}_{\mathscr{B}}P$, defined by means of the equivalence functor, makes $P_{\mathscr{A}}$ an essential $\mathscr{A}$-module, cf. the first paragraph of the proof.

Corollary 3.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras with bounded approximate identities. Then $\mathscr{A} \approx \mathscr{B}$, if and only if there are essential bimodules ${}_{\mathscr{B}}P_{\mathscr{A}}$ and ${}_{\mathscr{A}}Q_{\mathscr{B}}$ such that $Q\hat{\otimes}_{\mathscr{B}}P \cong \mathscr{A}$ and $P\hat{\otimes}_{\mathscr{A}}Q \cong \mathscr{B}$ as bimodules.*

Proof. The 'only if' part follows immediately. To prove the 'if' part let $F = {}_{\mathscr{B}}P\hat{\otimes}_{\mathscr{A}}-$ and let $G = {}_{\mathscr{A}}Q\hat{\otimes}_{\mathscr{B}}-$. Then $GF(M) = Q\hat{\otimes}_{\mathscr{B}}P\hat{\otimes}_{\mathscr{A}}M$. Defining $\tau_M = \pi_M\circ(\lambda\hat{\otimes}_{\mathscr{A}}\mathbf{1}_M)$ with $\pi_M:\mathscr{A}\hat{\otimes}_{\mathscr{A}}M \longrightarrow M$ the Rieffel isomorphism we get a natural isomorphism $\tau: GF \longrightarrow \mathbf{1}_{\mathrm{ess}-\mathscr{A}\text{-}\mathbf{mod}}$ with $\|\tau_M\| \le \|\lambda\|$ and $\|\tau_M^{-1}\| \le \|\lambda^{-1}\|d, d$ being the bound of the approximate identity. By symmetry $\mathscr{A} \approx \mathscr{B}$. $\square$

In the next corollary the superscript 'op' indicates the opposite Banach algebra and the tensor product of Banach algebras is defined as in [3], that is, if $\mathscr{A}$ and $\mathscr{B}$ are Banach algebras, then the product of $\mathscr{A}\hat{\otimes}\mathscr{B}$ is given by

$$(a_1\otimes b_1)(a_2\otimes b_2) = a_1a_2\otimes b_1b_2 \quad (a_i\in\mathscr{A}, b_i\in\mathscr{B}, i = 1,2).$$

Corollary 3.5. *Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ and $\mathscr{D}$ be Banach algebras and suppose that $\mathscr{A} \approx \mathscr{B}$ and $\mathscr{C} \approx \mathscr{D}$. Then $\mathscr{A}^{\mathrm{op}} \approx \mathscr{B}^{\mathrm{op}}$ and $\mathscr{A}\hat{\otimes}\mathscr{C} \approx \mathscr{B}\hat{\otimes}\mathscr{D}$. In particular there is a bounded, strongly continuous equivalence between the categories of essential bimodules* ess-$\mathscr{A}$-**mod**-$\mathscr{A}$ *and* ess-$\mathscr{B}$-**mod**-$\mathscr{B}$.

Proof. Let ${}_{\mathscr{B}}P_{\mathscr{A}}$, ${}_{\mathscr{A}}Q_{\mathscr{B}}$, ${}_{\mathscr{D}}V_{\mathscr{C}}$ and ${}_{\mathscr{C}}W_{\mathscr{D}}$ be modules providing the equivalences assumed. Defining ${}_{\mathscr{A}^{\mathrm{op}}}P_{\mathscr{B}^{\mathrm{op}}} = {}_{\mathscr{B}}P_{\mathscr{A}}$, etc. we see that $\mathscr{A}^{\mathrm{op}} \approx \mathscr{B}^{\mathrm{op}}$. Likewise, defining ${}_{\mathscr{B}\hat{\otimes}\mathscr{D}}X_{\mathscr{A}\hat{\otimes}\mathscr{C}} = {}_{\mathscr{B}}P_{\mathscr{A}}\hat{\otimes}{}_{\mathscr{D}}V_{\mathscr{C}}$ and ${}_{\mathscr{A}\hat{\otimes}\mathscr{C}}Y_{\mathscr{B}\hat{\otimes}\mathscr{D}} = {}_{\mathscr{A}}Q_{\mathscr{B}}\hat{\otimes}{}_{\mathscr{C}}W_{\mathscr{D}}$ with module multiplications given by $(b\otimes d).(p\otimes v) = bp\otimes dv$, etc., we see that the modules X and Y give a Morita equivalence $\mathscr{A}\hat{\otimes}\mathscr{C} \approx \mathscr{B}\hat{\otimes}\mathscr{D}$. $\square$

We have the following version of uniqueness of adjoints [20, IV Corollary 1.1]. We state it for covariant left adjoints.

Proposition 3.6. *Let* $F_1, F_2 : \text{ess-}\mathscr{A}\textbf{-mod} \longrightarrow \text{ess-}\mathscr{B}\textbf{-mod}$ *be bounded covariant functors. Suppose there is a bounded isomorphism*

$$I : {}_{\mathscr{B}}\mathbf{h}(F_1(M), N) \longrightarrow {}_{\mathscr{B}}\mathbf{h}(F_2(M), N),$$

natural in both variables. Then F_1 *and* F_2 *are naturally isomorphic.*

Proof. Define $\tau : F_2 \longrightarrow F_1$ by

$$\tau_M = I(\mathrm{id}_{F_1(M)}).$$

Then τ is easily seen to be natural with inverse $\tau_M^{-1} = I^{-1}(\mathrm{id}_{F_2(M)})$. Hence τ is a bounded natural isomorphism. □

Corollary 3.7. *Suppose F is an equivalence from* ess-$\mathscr{A}$-**mod** *to* ess-$\mathscr{B}$-**mod** *given by the bimodule* ${}_{\mathscr{B}}P_{\mathscr{A}}$. *Then the inverse is up to natural isomorphism* $\mathscr{A} \hat{\otimes}_{\mathscr{A}} {}_{\mathscr{B}}\mathbf{h}(P_{\mathscr{A}}, -)$.

Proof. $\mathscr{A} \hat{\otimes}_{\mathscr{A}} {}_{\mathscr{B}}\mathbf{h}(P_{\mathscr{A}}, -)$ is the right adjoint of ${}_{\mathscr{B}}P \hat{\otimes}_{\mathscr{A}} -$, see [12, II.5.22]. □

4. Morita contexts

Suppose two Banach algebras with bounded approximate identities, $\mathscr{A}$ and $\mathscr{B}$, are Morita equivalent, an let $\lambda : {}_{\mathscr{A}}Q \hat{\otimes}_{\mathscr{B}} P_{\mathscr{A}} \longrightarrow \mathscr{A}$ and $\mu : {}_{\mathscr{B}}P \hat{\otimes}_{\mathscr{A}} Q_{\mathscr{B}} \longrightarrow \mathscr{B}$ be the corresponding bimodule isomorphisms. Exactly as in [4, p. 33] we may adjust the isomorphism so that they are given by bounded balanced bilinear bimodule maps

$$(\cdot,\cdot) : Q \times P \longrightarrow \mathscr{A},$$

$$[\cdot,\cdot] : P \times Q \longrightarrow \mathscr{B},$$

satisfying

$$\begin{aligned} [p,q]p' &= p(q,p'), \\ q'[p,q] &= (q',p)q \end{aligned} \tag{1}$$

for all $p, p' \in P$, $q, q' \in Q$.

Definition 4.1. Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras with bounded approximate identities and let ${}_{\mathscr{B}}P_{\mathscr{A}}$ and ${}_{\mathscr{A}}Q_{\mathscr{B}}$ be essential bimodules. Suppose there are bounded module homomorphisms λ and μ given by bounded bilinear balanced module maps $(\cdot,\cdot) : Q \times P \longrightarrow \mathscr{A}$ and $[\cdot,\cdot] : P \times Q \longrightarrow \mathscr{B}$ satisfying (1) above. Then we call $(\mathscr{A}, \mathscr{B}, Q, P, \lambda, \mu)$ a *Morita context.* If λ and μ are epimorphisms, the Morita context is *full.*

Just as in the algebraic case we may now prove the following theorem.

Theorem 4.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras with bounded approximate identities. Then $\mathscr{A} \approx \mathscr{B}$ if and only if there is a full Morita context $(\mathscr{A}, \mathscr{B}, Q, P, \lambda, \mu)$.*

Proof. We have just noted that existence of a full Morita context, when $\mathscr{A} \approx \mathscr{B}$. Hence suppose we have a full Morita context. Then by definition P and Q are essential bimodules. We must show that the maps λ and μ are injective. Hence let $\sum (q_i, p_i) = 0$ and take $p \in P$, $q \in Q$. Then

$$(q,p)\sum q_i \otimes_{\mathscr{B}} p_i = \sum q[p,q_i] \otimes_{\mathscr{B}} p_i = \sum q \otimes_{\mathscr{B}} [p,q_i]p_i = q \otimes_{\mathscr{B}} p \sum (q_i,p_i) = 0$$

so $a\sum q_i \otimes_{\mathscr{B}} p_i = 0$ $\forall a \in \mathscr{A}$. Since $\mathscr{A}$ has a bounded approximate identity and $Q \hat{\otimes}_{\mathscr{B}} P$ is essential $\sum q_i \otimes_{\mathscr{B}} p_i = 0$. The Morita equivalence now follows from Corollary 3.4. □

Remark. Given a Morita context $(\mathscr{A}, \mathscr{B}, Q, P, \lambda, \mu)$, all we technically need in order to show that $\pi_M \circ (\lambda \otimes \mathbf{1}_M)$ is an isomorphism, is: There is a bounded approximate identity (e_i) for $\mathscr{A}$ and a bounded net (c_i) in $Q \hat{\otimes}_{\mathscr{B}} P$ such that $\lambda(c_i) = e_i$. However, this implies that λ is surjective. Any $a \in \mathscr{A}$ we may write as a Cohen factorization $a = a'x$. Looking at the proof of Cohen's factorization theorem, as it is for instance given in [3], we see that a' may be chosen of the form $a' = \sum_{k=1}^{\infty} \kappa_k e_{i_k}$ with (κ_k) a certain ℓ_1-sequence of complex numbers. Hence $a = \lambda(\sum \kappa_k c_{i_k}. x)$.

In [10, Theorem 5.4] it is proved that, if $\mathscr{A}$ is a Banach algebra with bounded approximate identity and $\mathscr{I}$ is a closed left ideal with a bounded left approximate identity for $\mathscr{A}$, then $\mathscr{A}$ is amenable if $\mathscr{I}$ is amenable. Presupposing Section 6, this follows from the following corollary.

Corollary 4.3. *Let $\mathscr{A}$ be a Banach algebra and $\mathscr{I}$ a closed left ideal, both having bounded approximate identities. Suppose that $\mathscr{I}$ in addition has a bounded left approximate identity for $\mathscr{A}$. Then $\mathscr{I} \approx \mathscr{A}$.*

Proof. With the natural product maps $\lambda : \mathscr{I} \hat{\otimes}_{\mathscr{I}} \mathscr{A} \longrightarrow \mathscr{A}$ and $\mu : \mathscr{A} \hat{\otimes}_{\mathscr{A}} \mathscr{I} \longrightarrow \mathscr{I}$ we see from Proposition 1.1 that $(\mathscr{A}, \mathscr{I}, {}_{\mathscr{A}}\mathscr{I}_{\mathscr{I}}, {}_{\mathscr{I}}\mathscr{A}_{\mathscr{A}}, \lambda, \mu)$ is a full Morita context. □

We now give the most important instance of a Morita context.

Example 4.4. Let $E \in$ **ess-mod-**$\mathscr{A}$, and let $E^{\#}$ be the essential part of $\mathbf{h}_{\mathscr{A}}(E, \mathscr{A}) \in \mathscr{A}$-**mod**, i.e. $E^{\#}$ consists of homomorphisms of the form $x \longrightarrow a\varphi(x)$ $(a \in \mathscr{A},$ $\varphi \in \mathbf{h}_{\mathscr{A}}(E, \mathscr{A}))$. Let $\mathscr{B}^{\#} = \mathbf{h}_{\mathscr{A}}(E, E)$. Then $E \in \mathscr{B}^{\#}$-**mod-**$\mathscr{A}$ and $E^{\#} \in \mathscr{A}$-**mod-**$\mathscr{B}^{\#}$. We define bilinear maps

$$(\cdot,\cdot) : E^{\#} \times E \longrightarrow \mathscr{A},$$

$$[\cdot,\cdot] : E \times E^{\#} \longrightarrow \mathscr{B}^{\#},$$

given by

$$(\varphi, x) = \varphi(x),$$

$$[x, \varphi](y) = x(\varphi, y)$$

for all $x, y \in E$, $\varphi \in E^{\#}$. With the canonical actions of $\mathscr{A}$ and $\mathscr{B}^{\#}$ $(\cdot,\cdot)$ is $\mathscr{B}^{\#}$-balanced and $[\cdot,\cdot]$ is $\mathscr{A}$-balanced, so that there is a Morita context $(\mathscr{A}, \mathscr{B}^{\#}, E^{\#}, E, \lambda, \mu)$. This is called the *derived* Morita context of E. Note that the image of μ is a two-sided ideal in $\mathscr{B}^{\#}$.

We now address the question: Given a Banach algebra $\mathscr{A}$ with bounded approximate identity, determine the Banach algebras with bounded approximate identities which are Morita equivalent to $\mathscr{A}$. Such an equivalence is by the Eilenberg–Watts theorem given by a certain module $P_{\mathscr{A}} \in$ ess-**mod**-$\mathscr{A}$ on which the algebra Morita equivalent to $\mathscr{A}$ acts on the left. We shall call such a module $P_{\mathscr{A}}$ an *equivalence module*, so more detailed the question can be phrased: Which modules in ess-**mod**-$\mathscr{A}$ are equivalence modules, and which Banach algebras Morita equivalent to $\mathscr{A}$ does a given equivalence module give rise to?

Before we proceed, we note a useful consequence of Cohen's factorization theorem.

Lemma 4.5. *Let $\varphi : \mathscr{C} \longrightarrow \mathscr{D}$ be a bounded algebra homomorphism between Banach algebras. Assume that $\mathscr{C}$ has a bounded left approximate identity and that $\varphi(\mathscr{C})$ is a right ideal. Then φ has closed range.*

Proof. Using φ we may regard $\mathscr{D}$ as a module in $\mathscr{C}$-**mod**. By Cohen's factorization theorem $\varphi(\mathscr{C})\mathscr{D}$ is closed. Now $\overline{\varphi(\mathscr{C})} = \overline{\varphi(\mathscr{C})\varphi(\mathscr{C})} \subseteq \overline{\varphi(\mathscr{C})\mathscr{D}} = \varphi(\mathscr{C})\mathscr{D} \subseteq \varphi(\mathscr{C})$. □

Theorem 4.6. *Let $\mathscr{A}$ and $\mathscr{B}$ be Banach algebras with bounded approximate identities, suppose that $\mathscr{A} \approx \mathscr{B}$, and let $P_{\mathscr{A}}$ be an equivalence module for this equivalence. Let $(\mathscr{A}, \mathscr{B}^{\#}, P^{\#}, P, \lambda, \mu)$ be the derived Morita context. Then μ maps $P \hat{\otimes}_{\mathscr{A}} P^{\#}$ isomorphically onto a closed two-sided ideal $\mathscr{E} \subseteq \mathscr{B}^{\#}$ with a bounded left approximate identity, and $\mathscr{B}$ is (isomorphic to) a closed left ideal of $\mathscr{E}$, such that $\mathscr{B}\mathscr{E} = \mathscr{E}$.*

Let $\mathscr{C}$ be any closed left ideal $\mathscr{C} \subseteq \mathscr{E}$ with a bounded approximate identity such that $\mathscr{C}\mathscr{E} = \mathscr{E}$, and put ${}_{\mathscr{A}}W_{\mathscr{C}} = \overline{P^{\#}.\mathscr{C}}$, the $\mathscr{A} - \mathscr{C}$ essential part of $P^{\#}$. Then the restricted Morita context $(\mathscr{A}, \mathscr{C}, W, P, \lambda, \mu)$ is full, so that $\mathscr{A} \approx \mathscr{C}$.

Proof. Let $(\mathscr{A}, \mathscr{B}, {}_{\mathscr{A}}Q_{\mathscr{B}}, {}_{\mathscr{B}}P_{\mathscr{A}}, \alpha, \beta)$ be a full Morita context establishing the equivalence, and let $(\cdot,\cdot)$ and $[\cdot,\cdot]$ be the corresponding balanced bilinear maps. Then we have a bounded module map $\Xi : {}_{\mathscr{A}}Q \longrightarrow {}_{\mathscr{A}}P^{\#}$ and a bounded algebra homomorphism $\Psi : \mathscr{B} \longrightarrow \mathscr{B}^{\#}$. These are given by $q \longrightarrow \xi$ and $b \longrightarrow \psi$ where

$$\xi(x) = (q, x), \quad x \in P$$

and

$$\psi(x) = b.x, \quad x \in P.$$

Here the module action of $\mathscr{B}$ on P is the given one. From the identity $\Psi \circ \beta = \mu \circ (1 \hat{\otimes}_{\mathscr{A}} \Xi)$ and surjectivity of β, it follows that $\Psi(\mathscr{B})$ is a left ideal in $\mathscr{B}^{\#}$. Since $\mathscr{B}$ has a bounded approximate identity, Ψ has closed range by Lemma 4.5. To see that Ψ is 1–1, suppose that $b.x = 0\ \forall x \in P$. Then $b[x,y] = [b.x, y] = 0\ \forall x \in P\ \forall y \in Q$ so that $bb' = 0\ \forall b' \in \mathscr{B}$. Since $\mathscr{B}$ has a bounded approximate identity, $b = 0$.

If we identify $\mathscr{B}$ with its image in $\mathscr{B}^{\#}$, the map Ξ is a morphism in $\mathscr{A}$-**mod**-$\mathscr{B}$. We now show that it is an isomorphism onto the $\mathscr{A} - \mathscr{B}$ essential part of $P^{\#}$. Clearly Ξ is a map into the essential part, since Q is essential. Suppose that $\xi = 0$, i.e. that $(q,x) = 0\ \forall x \in P$. Then as above we get that $q\beta(t) = 0\ \forall t \in P \hat{\otimes}_{\mathscr{A}} Q$ so that $q = 0$. Hence Ξ is injective. Now let $\xi.b$ be a generic element of the $\mathscr{B}$-essential part of $P^{\#}$, and write $b = \beta(t)$ with $t = \sum p_i \hat{\otimes}_{\mathscr{A}} q_i$. Then for $x \in P$ we have

$$\begin{aligned}\xi.b(x) &= \xi(\textstyle\sum [p_i, q_i]x) = \xi\big(\textstyle\sum p_i(q_i, x)\big)\\ &= \textstyle\sum \xi(p_i)(q_i, x)\\ &= \Xi\big(\textstyle\sum \xi(p_i) q_i\big)(x)\end{aligned}$$

so that $\xi.b = \Xi((\pi_Q \circ \xi \otimes_{\mathscr{A}} 1)(t))$, where $\pi_Q : \mathscr{A} \hat{\otimes}_{\mathscr{A}} Q \longrightarrow Q$ is the Rieffel isomorphism.

Let $x, y \in P$, let $q \in Q$, and let $\varphi \in P^{\#}$. Then $\mu([x,q]y \otimes_{\mathscr{A}} \varphi) = [x,q]\mu(y \otimes_{\mathscr{A}} \varphi)$, so that $\mathscr{E} = \mu(P \hat{\otimes}_{\mathscr{A}} P^{\#})$ is a two-sided ideal containing (the copy of) $\mathscr{B}$ and such that, using Cohen's factorization theorem, $\mathscr{B}\mathscr{E} = \mathscr{E}$. Arguing as in the proof of Lemma 4.5 we see that $\mathscr{E}$ is closed, and using that ${}_{\mathscr{B}}P$ is essential we may argue as above to show that μ is injective.

Finally, let $\mathscr{C} \subseteq \mathscr{E}$ be any such closed left ideal, and put ${}_{\mathscr{A}}W_{\mathscr{C}} = \overline{P^{\#}.\mathscr{C}}\ (\cong P^{\#} \hat{\otimes}_{\mathscr{C}} \mathscr{C})$. Since ${}_{\mathscr{B}}P$ is essential and $\mathscr{B} \subseteq \mathscr{E}$, P is essential as a left $\mathscr{E}$ module, and since $\mathscr{C}\mathscr{E} = \mathscr{E}$, it is essential as a left $\mathscr{C}$ module, when the module action of $\mathscr{E}$ is restricted to $\mathscr{C}$. It follows that $P \hat{\otimes}_{\mathscr{A}} (P^{\#}\mathscr{C})$ is mapped onto $\mathscr{C}$ and that $\varphi c \otimes_{\mathscr{C}} x \longrightarrow \varphi(cx) : (P^{\#}\mathscr{C}) \hat{\otimes}_{\mathscr{C}} P \longrightarrow \mathscr{A}$ is onto. Hence the Morita context $(\mathscr{A}, \mathscr{C}, W, P, \lambda, \mu)$ is full. □

Our next task is now to determine which modules $P_A \in$ ess-$\mathscr{A}$-**mod** give rise to full Morita contexts.

Proposition 4.7. *Let $P_A \in$ ess-$\mathscr{A}$-**mod**. Then the canonical map*

$$\lambda : {}_{\mathscr{A}}P^{\#} \hat{\otimes} P_{\mathscr{A}} \longrightarrow \mathscr{A}$$

is onto if and only if P_A is a generator.

Proof. A module is a generator if and only if it generates $\mathscr{A}$. Suppose the map λ is onto. Let $\bigoplus_\Gamma P_A$ be the direct sum of copies of P, where Γ is the unit ball of ${}_AP^{\#}$. If $\varphi : \bigoplus_\Gamma P_A \longrightarrow A$ is the map that lifts all $\gamma : P_A \longrightarrow A (\gamma \in \Gamma)$ then clearly φ is onto, when λ is onto. Conversely, suppose that $\mathscr{A}$ is a surjective image of a direct sum $\varphi : \bigoplus_\Gamma P_A \longrightarrow \mathscr{A}$. Each coordinate γ corresponds to an element of $\mathbf{h}_{\mathscr{A}}(P, \mathscr{A})$, namely

$\varphi_\gamma = \varphi \circ i_\gamma$. Since each element $a \in \mathscr{A}$ has the form $a = \sum_{\gamma \in \Gamma} \varphi_\gamma p_\gamma$ with $\Sigma \| p_\gamma \| < \infty$ we see that the map $\lambda : {}_{\mathscr{A}}P^\# \hat{\otimes} P_{\mathscr{A}} \longrightarrow \mathscr{A}$ is surjective. □

Let $\mathscr{B} = P \hat{\otimes}_{\mathscr{A}} P^\#$, and let as before $(\cdot,\cdot)$ and $[\cdot,\cdot]$ denote the bilinear maps from the derived Morita context. If we define a product

$$(x \otimes_{\mathscr{A}} \varphi)(y \otimes_{\mathscr{A}} \psi) = x(\varphi, y) \otimes_{\mathscr{A}} \psi \quad (x, y \in P;\ \varphi, \psi \in P^\#),$$

then $\mathscr{B}$ is a Banach algebra and $\mu : \mathscr{B} \longrightarrow \mathscr{B}^\#$ is a bounded algebra homomorphism mapping onto a (not necessarily closed) two-sided ideal of $\mathscr{B}^\#$.

We want to investigate when $\mathscr{B}$ has a bounded approximate identity. In order to give a symmetric treatment of left and right approximate identities we make the following definition.

Definition 4.8. Let $S \in \mathbf{h}_{\mathscr{A}}(P,P)$ then $S^\# \in {}_{\mathscr{A}}\mathbf{h}(P^\#, P^\#)$ is defined as

$$S^\# \varphi = \varphi S.$$

Proposition 4.9. *Let $(b_\gamma)_\Gamma$ be a bounded net in $P \hat{\otimes}_{\mathscr{A}} \mathscr{P}^\#$. Suppose that $P_{\mathscr{A}}$ is a generator. Then $(b_\gamma)_\Gamma$ is a left bounded approximate identity if and only if $\mu(b_\gamma) \longrightarrow \mathrm{id}_P$ b.s.o. and $(b_\gamma)_\Gamma$ is a right bounded approximate identity if and only if $\mu(b_\gamma)^\# \longrightarrow \mathrm{id}_{P^\#}$ b.s.o.*

Proof. From the identities

$$b(x \otimes_{\mathscr{A}} \varphi) = \mu(b)(x) \otimes_{\mathscr{A}} \varphi$$

and

$$(x \otimes_{\mathscr{A}} \varphi) b = x \otimes_{\mathscr{A}} \mu(b)^\# \varphi$$

we see that b.s.o. convergence implies the existence of approximate identities. Conversely, since $P_{\mathscr{A}}$ is essential generator we get for a generic element $y(\varphi, x) \in P_{\mathscr{A}}$ that $\mu(b_\gamma)(y(\varphi, x)) = \mu(b_\gamma y \otimes_{\mathscr{A}} \varphi)(x) \longrightarrow [y, \varphi] x = y(\varphi, x)$, so that $\mu(b_\gamma) \longrightarrow \mathrm{id}_P$ b.s.o. when $(b_\gamma)_\Gamma$ is a bounded left approximate identity. Similarly, let $(\varphi, y)\psi$ be a generic element in $P^\#$. Then

$$\begin{aligned}\mu(b_\gamma)^\#((\varphi, y)\psi) &= \mu(b_\gamma)^\#(\varphi[y, \psi]) = \varphi[y, \psi]\mu(b_\gamma) \\ &= \varphi\mu((y \otimes_{\mathscr{A}} \psi) b_\gamma) \to \varphi[y, \psi] = (\varphi, y)\psi,\end{aligned}$$

when $(b_\gamma)_\Gamma$ is a right approximate identity. □

We now give a criterion for $\mathscr{B}$ to have a bounded left approximate identity.

Definition 4.10. Let $P \in \text{ess-}\mathbf{mod}\text{-}\mathscr{A}$. We say that P is a *progenerator* if P is a generator and there are finite sequences $(x_i)_1^n$ in P and $(\varepsilon_i)_1^n$ in $\mathbf{h}_{\mathscr{A}}(P, A)$ such that $p = \sum_1^n x_i \varepsilon_i(p)$ for all $p \in P$. We say that P is an *approximate progenerator* if P is a generator and there are nets of sequences $((y_{n\gamma})_{\mathbb{N}})_\Gamma$ in P and $((\varepsilon_{n\gamma})_{\mathbb{N}})_\Gamma$ in $\mathbf{h}_{\mathscr{A}}(P, A)$ with $\|y_{n\gamma}\| = 1$ $(n \in \mathbb{N}, \gamma \in \Gamma)b$ and $\sup_\gamma\{\sum_n \|\varepsilon_{n\gamma}\|\} < \infty$ such that

$$\lim_\Gamma \sum y_{n\gamma}\varepsilon_{n\gamma}(p) = p \quad (p \in P).$$

Note that we may state the condition for a progenerator as the existence of a finite set of projective coordinates. Hence a progenerator is a finitely generated projective generator (exactly as it is defined in the purely algebraic case). Similarly an approximate progenerator is approximately projective. Note also that, since $P_{\mathscr{A}}$ is essential we may replace $\mathbf{h}_{\mathscr{A}}(P, \mathscr{A})$ by $P^\#$.

Proposition 4.11. *Let $P_{\mathscr{A}}$ be a generator and consider the derived Morita context: $(\mathscr{A}, \mathscr{B}^\#, P^\#, P, \lambda, \mu)$. Let $\mathscr{B} = P \hat{\otimes}_{\mathscr{A}} P^\#$. Then $\mathscr{B}$ is unital if and only if $P_{\mathscr{A}}$ is a progenerator, in which case $\mu : \mathscr{B} \longrightarrow \mathscr{B}^\#$ is an isomorphism. $\mathscr{B}$ has a bounded left approximate identity if and only if $P_{\mathscr{A}}$ is an approximate progenerator, in which case $\mu : \mathscr{B} \longrightarrow \mathscr{B}^\#$ is an isomorphism onto its range.*

Proof. By (the proof of) Proposition 4.9 the Banach algebra $\mathscr{B}$ is left unital if and only if $\exists b \in \mathscr{B}$ such that $\mu(b) = \mathrm{id}_P$. But then clearly $\mu(b)^\# = \mathrm{id}_{P^\#}$ so that $\mathscr{B}$ is actually unital. If $P_{\mathscr{A}}$ is a progenerator with projective coordinates $\varepsilon_1, \ldots, \varepsilon_n$ and generators $x_1, \ldots, x_n$, then $\mu(\sum_1^n x_i \hat{\otimes}_{\mathscr{A}} \varepsilon_i) = \mathrm{id}_P$. Conversely, suppose that $\mu(\sum_1^\infty x_i \otimes_{\mathscr{A}} \varepsilon_i) = \mathrm{id}_P$ with $\sum_1^\infty \|x_i\| \|\varepsilon_i\| < \infty$. Then for n sufficiently large the element $S = \mu(\sum_1^n x_i \otimes_{\mathscr{A}} \varepsilon_i)$ in $B^\#$ is an invertible operator. It follows that $\varepsilon_1 S^{-1}, \ldots, \varepsilon_n S^{-1}$ and $x_1, \ldots, x_n$ constitute a finite system of projective coordinates and generators.

Similarly $\mathscr{B}$ has a left approximate identity if and only if there is a bounded net $(b_\gamma)_\Gamma$ so that $\mu(b_\gamma) \longrightarrow \mathrm{id}_P$ b.s.o. But, as above, this means that $P_{\mathscr{A}}$ is an approximate progenerator. If $P_{\mathscr{A}}$ is an approximate progenerator, then μ has closed range by Lemma 4.5. Then injectivity follows as in the proof of Theorem 4.2. □

Combining Proposition 4.11 with Theorem 4.6 we get (with $\mathscr{C}$ and $\mathscr{E}$ as in the notation of Theorem 4.6) the following corollary.

Corollary 4.12. *Let $P_{\mathscr{A}}$ be an equivalence module. Then $P_{\mathscr{A}}$ is an approximate progenerator. Conversely, if $P_{\mathscr{A}}$ is an approximate progenerator, then $P_{\mathscr{A}}$ is an equivalence module for an equivalence with any $\mathscr{C} \subseteq \mathscr{E}$ (if any) satisfying the hypothesis in Theorem 4.6.*

Corollary 4.13. *Let $\mathscr{A} \approx \mathscr{B}$ and suppose that $\mathscr{B}$ is unital. Then there are $n \in \mathbb{N}$ and an idempotent matrix $p \in M_n(\mathscr{A})$ so that $\mathscr{B} \cong pM_n(\mathscr{A})p$.*

Corollary 4.14. *Let p be an idempotent in $\mathscr{M}(\mathscr{A})$, the double centralizer algebra of $\mathscr{A}$. Assume that the multiplication map*

$$\mathscr{A}p \hat{\otimes} p\mathscr{A} \longrightarrow \mathscr{A}$$

is surjective. Then $p\mathscr{A}p \approx \mathscr{A}$. In particular, $\mathscr{A} \approx M_n(\mathscr{A})$.

Example 4.15. Which Banach algebras (with bounded approximate identities) are Morita equivalent to $\mathbb{C}$? Let X be a Banach space such that $X_{\mathbb{C}}$ is an equivalence module. Then $X_{\mathbb{C}}$ is finite-dimensional. To see this we form the derived Morita context $(\mathbb{C}, \mathscr{B}(X), X^*, X, \mathrm{tr}, \mathrm{Tr})$ where $\mathrm{tr}(x^* \otimes x) = x^*(x)$ and $\mathrm{Tr}(x \otimes x^*)$ is the rank 1 operator $\xi \longrightarrow x^*(\xi)x$. From Theorem 4.12 it follows that the tensor algebra $X \hat{\otimes} X^*$ has a bounded left approximate identity. But then X is finite dimensional, see [32, Corollary 2.6]. Thus the only Banach algebras Morita equivalent to $\mathbb{C}$ are the matrix algebras $M_n(\mathbb{C})$, exactly as in the discrete case.

5. Elementary Morita invariants

In this chapter we treat properties which are rather easily seen to be preserved by Morita equivalence. Throughout F and G is a pair of equivalence functors and ϕ and θ are the associated natural isomorphisms for adjointness as in Section 1. Our exposition only adds the necessary functional analytic overtones to standard treatments such as [1].

Theorem 5.1. *Let $\mathscr{A} \approx \mathscr{B}$ and let F be an equivalence functor. Let $M \in$ ess-$\mathscr{A}$-**mod**. Then*

(1) *M is injective (projective, approximately projective) $\Leftrightarrow$ FM is injective (projective, approximately projective).*
(2) *M is a generator (cogenerator, faithful) $\Leftrightarrow$ FM is a generator (co-generator, faithful).*
(3) *M is (topologically) irreducible $\Leftrightarrow$ FM is (topologically) irreducible.*

Proof. (1) The only functional analytic considerations concern the statement about approximately projective modules. Consider the test diagram

$$\begin{tikzcd} & FM \arrow[d, "\varphi"] & \\ Y \arrow[r, "q"] & Z \arrow[r] & 0 \end{tikzcd}$$

and apply G and θ to get a test diagram

$$\begin{tikzcd} & M \arrow[d, "\theta\varphi"] & \\ GY \arrow[r, "Gq"'] & GZ \arrow[r] & 0 \end{tikzcd}$$

Assuming that M is approximately projective there is a bounded family $\psi_i \in {}_{\mathscr{A}}\mathbf{h}(M, GY)$ such that $(Gq) \circ \psi_i \longrightarrow \theta\varphi$. Since θ is an isomorphism with respect to b.s.o. topologies we get that $\theta^{-1}(Gq \circ \psi_i) = q\theta^{-1}(\psi_i) \longrightarrow \varphi$ b.s.o., completing the original test diagram. Conversely, if FM is approximately projective then GFM, and hence its isomorphic image M is approximately projective.

(2) This is immediate, using Proposition 1.9, since an equivalence preserves direct sums and products and ϕ and θ preserve epimorphisms and embeddings.

(3) This follows from Corollary 2.10 together with M being (topologically) irreducible if and only if for all $M' \in$ ess-$\mathscr{A}$-**mod** the set ${}_{\mathscr{A}}\mathbf{h}(M', M)$ consists of 0 and (epics) epimorphisms. □

Corollary 5.2. *Suppose $\mathscr{A} \approx \mathscr{B}$. Then $\mathscr{A}$ is primitive (semisimple), if and only if $\mathscr{B}$ is primitive (semisimple).*

Proof. $\mathscr{A}$ is primitive if and only if it has a faithful irreducible module. $\mathscr{A}$ is semisimple if and only if there is a faithful module, which is a direct sum of irreducible modules. □

One of the most important features of an equivalence functor is that it takes the lattice of closed submodules of given module to the lattice of closed submodules of the equivalent module. Using the open mapping theorem, the purely algebraic proof of this can be adapted to give the next theorem. For a Banach module M, denote the lattice of closed submodules of M, by $\mathrm{Lat}(M)$. For two modules $K \subseteq M$ we let $\iota_{K \to M}$ denote the inclusion.

Theorem 5.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be Morita equivalent with equivalence functor F. Then for each $M \in$ ess-$\mathscr{A}$-**mod** the map*

$$\Lambda_M(K) = \operatorname{im} F(\iota_{K \to M})$$

is a lattice isomorphism from $\mathrm{Lat}(M)$ *onto* $\mathrm{Lat}(FM)$.

Proof. Modulo applications of the open mapping theorem, this follows closely the algebraic proof. Let K be a submodule of M. Since an equivalence functor preserves short exact sequences $\Lambda_M(K)$ is a closed submodule so that Λ_M: $\mathrm{Lat}(M) \longrightarrow \mathrm{Lat}(FM)$. Since F is a functor, Λ_M is order preserving. We claim that

$$\Gamma_M(N) = \operatorname{im} \phi(\iota_{N \to FM})$$

is the lattice inverse to Λ_M. Firstly, by Corollary 2.10, $\Gamma_M(N)$ is a closed submodule of M and since ϕ is natural, that is, $\phi(i_2 i_1) = \phi(\mathrm{i}_2)\mathrm{G}i_1$, Γ_M is order preserving. Let $N = \Lambda_M(K)$ and consider

$$\begin{tikzcd} N \arrow[r, "\iota_{N\to FM}"] & FM \\ & FK \arrow[u, "F(\iota_{K\to M})"'] \end{tikzcd}$$

Since $F(\iota_{K\to M})$ is an isomorphism of FK onto $\Lambda_M(K) = N$ there is, by the open mapping theorem, an isomorphism $h: FK \longrightarrow N$, such that $\iota_{N\to FM}\, h = F(\iota_{K\to M})$. Applying ϕ we get

$$\phi(\iota_{N\to FM})\, Gh = \phi(F(\iota_{K\to M})) = \iota_{K\to M}\,\phi(\mathrm{id}_{FK}).$$

Since Gh and $\phi(\mathrm{id}_{FK})$ are isomorphisms onto the domains of $\phi(\iota_{N\to FM})$ and $\iota_{K\to N}$, respectively, we see that $\operatorname{im}\phi(\iota_{N\to FM}) = \operatorname{im}\iota_{K\to M} = K$. Now let $N \subseteq FM$ and put $K = \Gamma_M(N)$. Then look at

$$\begin{tikzcd} GN \arrow[dr, "\phi(\iota_{N\to FM})"] & \\ K \arrow[r, "\iota_{K\to M}"'] & M \end{tikzcd}$$

Since $\phi(\iota_{N\to FM})$ has closed range there is, by the open mapping theorem, an isomorphism $\gamma: GN \longrightarrow K$, such that

$$\phi(\iota_{N\to FM}) = \iota_{K\to M}\circ\gamma .$$

Hence

$$\iota_{N\to FM} = \phi^{-1}(\iota_{K\to M}\circ\gamma) = F(\iota_{K\to M})\,\phi^{-1}(\gamma)$$

so that

$$\Lambda_M(K) = \operatorname{im} F(\iota_{K\to M}) = \operatorname{im}\iota_{N\to FM} = N . \quad \square$$

Next we prove that equivalent Banach algebras have isomorphic centres. The nomenclature needed is the following:

Definition 5.4. The set of bounded natural transformations from the identity functor $\mathbf{1}_{\text{ess-}\mathscr{A}\text{-mod}}$ to itself is called $\mathrm{Nat}(\mathscr{A})$. It is a commutative Banach algebra with composition as product and norm as in Definition 2.2. The centre of $\mathscr{A}$ (sometimes also called the ideal centre) is the commutative Banach algebra $\mathrm{Cen}(\mathscr{A}) = \{\varphi \in \mathscr{L}(\mathscr{A}) \mid \varphi R_a = R_a\varphi \ \forall a \in \mathscr{A}\} = \{\varphi \in \mathscr{R}(\mathscr{A}) \mid \varphi L_a = L_a\varphi \forall a \in \mathscr{A}\} = {}_{\mathscr{A}}\mathbf{h}_{\mathscr{A}}(\mathscr{A}, \mathscr{A})$.

Proposition 5.5. Cen($\mathscr{A}$) *and* Nat($\mathscr{A}$) *are isomorphic.*

Proof. Let $\varphi \in \mathrm{Cen}(\mathscr{A})$ and let $M \in$ ess-$\mathscr{A}$-**mod**. Then $M \in \mathscr{L}(\mathscr{A})$-**mod** by Proposition 1.10, so the map $\eta_M : m \longrightarrow \varphi . m$ defines a bounded natural transformation. Conversely, if $\eta : \mathbf{1}_{\text{ess-}\mathscr{A}\text{-mod}} \longrightarrow \mathbf{1}_{\text{ess-}\mathscr{A}\text{-mod}}$ is a bounded natural transformation, then $\eta_A \in \mathrm{Cen}(\mathscr{A})$. □

Theorem 5.6. *If* $\mathscr{A} \approx \mathscr{B}$, *then* Cen($\mathscr{A}$) *and* Cen($\mathscr{B}$) *are isomorphic as Banach algebras.*

Proof. We invoke Proposition 5.5. Let F, G, η and ζ be as usual and let $\mathrm{Nat}(\mathscr{A}) \underset{S}{\overset{T}{\rightleftarrows}} \mathrm{Nat}(\mathscr{B})$ be defined by

$$T(\sigma)_N = G^{-1}(\sigma_{GN}) = \zeta_N F(\sigma_{GN}) \zeta_N^{-1},$$

$$S(\tau)_M = F^{-1}(\tau_{FM}) = \eta_M G(\tau_{FM}) \eta_M^{-1},$$

for M in ess-$\mathscr{A}$-**mod** and N in ess-$\mathscr{B}$-**mod**. Then

$$ST(\sigma)_M = F^{-1}((T\sigma)_{FM}) = F^{-1}(\zeta_{FM} F(\sigma)_{GFM} \zeta_{FM}^{-1})$$

By Proposition 2.4 there is an isomorphism $\alpha : GFM \longrightarrow M$ such that $F(\alpha) = \zeta_{FM}$. Hence $ST(\sigma)_M = \alpha \sigma_{GFM} \alpha^{-1} = \sigma_M$, where the last equality follows from θ being natural. That $T(\sigma)$ is a natural transformation is seen by noting that a diagram is commutative if and only if its image under G is commutative. □

It is now clear that the theory for Morita equivalence is a theory for non-commutative Banach algebras in the sense that, even for commutative algebras, non-trivial results involve non-commutative algebras.

Corollary 5.7. *Two unital commutative Banach algebras are Morita equivalent if and only if they are isomorphic. A unital Banach algebra is Morita equivalent to a commutative Banach algebra if and only if it is equivalent to its own centre.*

Now let $\mathscr{A} \approx \mathscr{B}$ via an equivalence F: ess-$\mathscr{A}$-**mod** $\longrightarrow$ ess-$\mathscr{B}$-**mod**. Then for each two-sided closed ideal in $\mathscr{A}$, the set

$$\Phi(I) = \mathrm{ann}_{\mathscr{B}}(F(\mathscr{A}/I))$$

is a closed two-sided ideal in $\mathscr{B}$. The map Φ is actually a lattice isomorphism.

Theorem 5.8. *If* $\mathscr{A} \approx \mathscr{B}$, *then the mapping* Φ *is a lattice isomorphism from the lattice of closed two-sided ideals in* $\mathscr{A}$ *onto the lattice of closed two-sided ideals in* $\mathscr{B}$. *Furthermore, for each closed two-sided ideal in* $\mathscr{A}$, *there is an equivalence*

$$\mathscr{A}/I \approx \mathscr{B}/\Phi(I).$$

Proof. Except for some obvious changes, this can be proved exactly as it is proved in [1, 21.11]. □

Corollary 5.9. *If $\mathscr{A}$ is topologically simple, and $\mathscr{A} \approx \mathscr{B}$, then $\mathscr{B}$ is topologically simple.*

Corollary 5.10. *If $\mathscr{A} \approx \mathscr{B}$, then their structure spaces $\Pi_{\mathscr{A}}$ and $\Pi_{\mathscr{B}}$ are homeomorphic when equipped with the hull-kernel topologies. Furthermore $\mathscr{A}/\operatorname{rad}\mathscr{A} \approx \mathscr{B}/\operatorname{rad}\mathscr{B}$.*

Proof. From Corollary 5.2 we see that an ideal $P \in \Pi_{\mathscr{A}}$ if and only if $\Phi(P) \in \Pi_{\mathscr{B}}$. Since Φ is a lattice isomorphism we get for $E \subseteq \Pi_{\mathscr{A}}$ that $\Phi(\ker E) = \Phi(\bigcap\{P \mid P \in E\}) = \bigcap\{\Phi(P) \mid P \in E\} = \ker \Phi(E)$. Similarly for a closed two-sided ideal $\mathscr{I} \subseteq \mathscr{A}$ we get $\operatorname{hull}\Phi(\mathscr{I}) = \Phi(\operatorname{hull}\mathscr{I})$. In particular $\Phi(\operatorname{rad}(\mathscr{A})) = \operatorname{rad}(\mathscr{B})$ whence, invoking Theorem 5.8, $\mathscr{A}/\operatorname{rad}\mathscr{A} \approx \mathscr{B}/\operatorname{rad}\mathscr{B}$. □

6. Morita invariance of bounded Hochschild homology

Let us assume $\mathscr{A} \approx \mathscr{B}$. By Corollary 3.6 there is an equivalence between the categories of essential bimodules given by $P \in$ ess-$\mathscr{B}$-**mod**-$\mathscr{A}$ and $Q \in$ ess-$\mathscr{A}$-**mod**-$\mathscr{B}$ so that the equivalence functor

$$\text{ess-}\mathscr{A}\text{-}\mathbf{mod}\text{-}\mathscr{A} \longrightarrow \text{ess-}\mathscr{B}\text{-}\mathbf{mod}\text{-}\mathscr{B}$$

has the form

$$M \longrightarrow P \mathbin{\hat{\otimes}_{\mathscr{A}}} M \mathbin{\hat{\otimes}_{\mathscr{A}}} Q.$$

In particular the modules P and Q are flat as $\mathscr{A}$- and $\mathscr{B}$-modules.

We want to show that $\mathscr{H}_n(\mathscr{A}, M) \cong \mathscr{H}_n(\mathscr{B}, P \mathbin{\hat{\otimes}_{\mathscr{A}}} M \mathbin{\hat{\otimes}_{\mathscr{A}}} Q)$, where the isomorphism is an isomorphism of semi-normed spaces. Our motivation to do this originates in questions of computing Hochschild cohomology with coefficients in dual modules, that is, the theory obtained by dualizing Hochshild homology. In pure algebra Morita invariance of Hochschild cohomology for unital algebras is easily treated by viewing Hochschild cohomology groups as cases of Ext-groups and utilizing that an equivalence preserves projective resolutions and hom-spaces. There are two reasons that this will not work in our case. First of all there may be no projective objects in ess-$\mathscr{A}$-**mod**-$\mathscr{A}$. The second problem arises in the definition of Ext in the Banach algebra case. We want resolutions to split as Banach spaces and such a splitting is *not* preserved by the equivalence. We shall therefore use a different approach and adapt the proof of [7]. The idea, due to F. Waldhausen, is to make a double complex (in the first quadrant) starting by placing the chain complex giving the Hochschild homology for $\mathscr{A}$ on the first axis and the chain complex giving the Hochschild homology for $\mathscr{B}$ on the second axis and then fill out so that the double complex is vertically and horizontally acyclic except possibly on the axes. A standard spectral sequence argument will then in the algebraic case finish the proof. However, rather than developing spectral sequence techniques in general for double complexes of Banach spaces and bounded chain differentials we give an ad hoc argument for the case we need:

Lemma 6.1. *Let $M_{p,q}$ be a double complex of Banach spaces in the first or third quadrant with bounded differentials $d'_{p,q}: M_{p,q} \longrightarrow M_{p-1,q}$ and $d''_{p,q}: M_{p,q} \longrightarrow M_{p,q-1}$. Assume that $\mathscr{H}_*(M_{*,q}, d'_{*,q}) = \mathscr{H}_*(M_{p,*}, d''_{p,*}) = \{0\}$ for $q \neq 0$ and $p \neq 0$ (i.e. rows and columns are acyclic except possibly on the axes). Then $\mathscr{H}_*(M_{*,0}, d'_{*,0})$ is isomorphic as a graded semi-normed space to $\mathscr{H}_*(M_{0,*}, d''_{0,*})$.*

Proof. We give the proof for a double complex in the first quadrant. Denote the vertical complexes $(M_{p,q}, d''_{p,q})$ by (C^p, d^p) $p \geq 0$. Assume that (C^p, d^p) is acyclic for $p \geq 1$. Recall that $(\operatorname{Tot} M, d)$ is the complex given by $(\operatorname{Tot} M)_n = \bigoplus_{p+q=n} M_{p,q}$ and $d_n = \bigoplus_{p+q=n} d'_{p,q} + d''_{pq}$. The natural embedding $(C^0, d^0) \longrightarrow (\operatorname{Tot} M, d)$ is a chain map, so we have a long exact sequence

$$\mathscr{H}_*(C^0, d^0) \longrightarrow \mathscr{H}_*(\operatorname{Tot} M, d) \longrightarrow \mathscr{H}_*(\operatorname{Tot} M/C^0, d) \longrightarrow \mathscr{H}_{*-1}(C^0, d^0)$$

Here $(\operatorname{Tot} M/C^0, d)$ is the natural quotient chain complex. This complex is (isomorphic to) the total complex of the double complex M^1 obtained from M by replacing the 0th column by the zero-column, i.e. $M^1_{p,q} = M_{p,q}$ for $p \geq 1$, $M^1_{0,q} = \{0\}$, and $d'_{p,q}$ for $p \geq 2$, $d''_{p,q}$ for $p \geq 1$ are unchanged. Similarly we define M^n to be the double complex obtained from M be replacing the 0th – $(n-1)$th columns by zero-columns. As above, we get long exact sequences

$$\mathscr{H}_*(C^p, d^p) \longrightarrow \mathscr{H}_*(\operatorname{Tot} M^p, d) \longrightarrow \mathscr{H}_*(\operatorname{Tot} M^{p+1}, d) \longrightarrow \mathscr{H}_{*-1}(C^p, d^p).$$

Now for $p \geq 1$, $\mathscr{H}_*(C^p, d^p) = (0)$ so we have

$$\mathscr{H}_*(\operatorname{Tot} M^p, d) \cong \mathscr{H}_*(\operatorname{Tot} M^{p+1}, d)$$

for $p \geq 1$. Since M^p is supported to the right of p we have $\mathscr{H}_n(\operatorname{Tot} M^p, d) = (0)$ for $p > n$. It follows that $\mathscr{H}_*(\operatorname{Tot} M^1, d) = (0)$, so that $\mathscr{H}_*(C^0, d^0) \cong \mathscr{H}_*(\operatorname{Tot} M, d)$. Since the homology groups have the form $\ker T/\operatorname{im} S$, where T and S are bounded linear maps, they are semi-normed spaces. The maps in the long exact sequences are continuous maps of semi-normed spaces, so the above isomorphism is topological. Similarly, we get that $\mathscr{H}_*(M_{*,0}, d'_{*,0}) \cong \mathscr{H}_*(\operatorname{Tot} M, d)$, proving the claim. □

We now state the theorem on Morita invariance of bounded Hochschild homology. In order to make a symmetric statement, let $N = P \hat{\otimes}_{\mathscr{A}} M$. Then $M \cong Q \hat{\otimes}_{\mathscr{B}} N$ and $P \hat{\otimes}_{\mathscr{A}} M \hat{\otimes}_{\mathscr{A}} Q \cong N \hat{\otimes}_{\mathscr{A}} Q$.

Theorem 6.2. *With notation as above we have*

$$\mathscr{H}_*(\mathscr{A}, Q \hat{\otimes}_{\mathscr{B}} N) \cong \mathscr{H}_*(\mathscr{B}, N \hat{\otimes}_{\mathscr{A}} Q)$$

as semi-normed spaces.

Proof. We start by forming a double complex with the Hochschild complex for $\mathscr{A}$, shifted 1, on the horizontal axis and the Hochschild complex for $\mathscr{B}$, shifted 1, on the vertical axis. Thus

$$C_{0n} = C_{n-1}(\mathscr{B}, N \hat{\otimes}_{\mathscr{A}} Q) = \mathscr{B}^{\hat{\otimes}(n-1)} \hat{\otimes} N \hat{\otimes}_{\mathscr{A}} Q,$$

$$C_{m0} = C_{m-1}(\mathscr{A}, Q \hat{\otimes}_{\mathscr{B}} N) = \mathscr{A}^{\hat{\otimes}(m-1)} \hat{\otimes} Q \hat{\otimes}_{\mathscr{B}} N,$$

$$C_{00} = (0)$$

We then fill out the rest of the first quadrant by

$$\begin{aligned} C_{mn} &= \mathscr{A}^{\hat{\otimes}(m-1)} \hat{\otimes} Q \hat{\otimes} \mathscr{B}^{\hat{\otimes}(n-1)} \hat{\otimes} N \\ &= C_{m-1}(\mathscr{A}, Q \hat{\otimes} \mathscr{B}^{\hat{\otimes}(n-1)} \hat{\otimes} N) \\ &= C_{n-1}(\mathscr{B}, N \hat{\otimes} \mathscr{A}^{\hat{\otimes}(m-1)} \hat{\otimes} Q), \end{aligned}$$

where the identifications made just are permutations of the tensor factors. We next define vertical and horizontal differentials to form the double complex. We let $d_h: C_{mn} \longrightarrow C_{(m-1)n}$ $(m \geq 2, n \geq 0)$ be the Hochschild differential ∂_{m-1}: $C_{m-1}(\mathscr{A}, Q \hat{\otimes} \mathscr{B}^{\hat{\otimes}(n-1)} \hat{\otimes} N) \longrightarrow C_{m-2}(\mathscr{A}, Q \hat{\otimes} \mathscr{B}^{\hat{\otimes}(n-1)} \hat{\otimes} N)$, where formally $\mathscr{B}^{\hat{\otimes}(0)} = \mathbb{C}$ and $Q \hat{\otimes} \mathscr{B}^{\hat{\otimes}(-1)} \hat{\otimes} N = Q \hat{\otimes}_{\mathscr{B}} N$. We define $d_h: C_{1n} \longrightarrow C_{0n}$ for $n \geq 1$ by

$$d_h(q \otimes b^{(n-1)} \otimes x) = b^{(n-1)} \otimes x \otimes_{\mathscr{A}} q,$$

where $b^{(n-1)} \in \mathscr{B}^{\hat{\otimes}(n-1)}$, $q \in Q$ and $x \in N$.

The vertical differentials are defined by interchanging the rôles of $\mathscr{A}$ and $\mathscr{B}$. Since the horizontal and vertical differentials "operate" at disjoint places, it is clear that they commute, so if we alternate signs on, say the vertical, differentials we have a double complex.

We now show that the row complexes are acyclic for $m \geq 1$. By definition of $\hat{\otimes}_{\mathscr{A}}$ the maps $d_h: C_{1n} \longrightarrow C_{0n}$ $(n \geq 1)$ are onto. It is also clear that $d_h \circ d_h = 0$ at $m = 1$. The acyclicity now follows from the following two lemmas, whose proofs in essence are adaptations of the proof of H-unitality of Banach algebras with bounded approximate identities, [36, Proposition 5.1.].

Lemma 6.3. *The horizontal complexes* $\cdots \longrightarrow C_{2n} \longrightarrow C_{1n} \longrightarrow C_{0n} \longrightarrow 0$ $(n \geq 1)$ *are exact for* $m = 0$ *and* $m = 1$.

Proof. We have already noted the exactness at $m = 0$. Now

$$d_h : \mathscr{B}^{\hat{\otimes}(n-1)} \hat{\otimes} \mathscr{A} \hat{\otimes} Q \hat{\otimes} N \longrightarrow \mathscr{B}^{\hat{\otimes}(m-1)} \hat{\otimes} Q \hat{\otimes} N$$

is given by

$$b^{(n-1)} \otimes a \otimes q \otimes x \longrightarrow b^{(n-1)} \otimes q \otimes x.a - b^{(n-1)} \otimes a.q \otimes x,$$

where $b^{(n-1)} \in \mathscr{B}^{\hat{\otimes}(n-1)}$, $a \in \mathscr{A}$, $q \in Q$ and $x \in N$. Suppose that we have a cycle in C_{1n}, that is, $\sum_{i=1}^{\infty} b_i^{(n-1)} \otimes x_i \otimes_{\mathscr{A}} q_i = 0$ with $\sum_{i=1}^{\infty} \|b_i^{(n-1)}\| \, \|x_i\| \, \|q_i\| < \infty$. Let $Z \in$ ess-**mod**-$\mathscr{B}$ be the module of sequences in Q of the form (ζ_i), where $\|(\zeta_i)\| \equiv \sum \|b_i^{(n-1)}\| \, \|x_i\| \, \|\zeta_i\| < \infty$. Since $(q_i) \in Z$ we may, by Cohen's factorization theorem, write $(q_i) = (\zeta_i).b$ with $(\zeta_i) \in ((q_i).\mathscr{B})^{-} \subseteq Z$. Let $Q \hat{\otimes}_{\mathscr{B}} P \cong \mathscr{A}$ and $P \hat{\otimes}_{\mathscr{A}} Q \cong \mathscr{B}$ be given by bimodule isomorphisms λ and μ respectively as in chapter 4, and write $b = \mu(\sum p_j \otimes_{\mathscr{A}} r_j)$. Put $a_{ij} = \lambda(\zeta_i \otimes_{\mathscr{B}} p_j)$ and let

$$u = -\sum_{i,j=1}^{\infty} b_i^{(n-1)} \otimes a_{ij} \otimes r_j \otimes x_i.$$

Then

$$d_h(u) = \sum_{i,j} b_i^{(n-1)} \otimes a_{ij}.r_j \otimes x_i - \sum_{i,j} b_i^{(n-1)} \otimes r_j \otimes x_i.a_{ij}.$$

Let $b^{(n-1)} \in \mathscr{B}^{\otimes(n-1)}$ and $p \in P$. Then the map from $N \times Q$ defined by

$$(x, q) \longrightarrow b^{(n-1)} \otimes x.\lambda(q \otimes_{\mathscr{B}} p)$$

is clearly $\mathscr{A}$-balanced, so, since $\sum_{i=1}^{\infty} b_i^{(n-1)} \otimes x_i \otimes_{\mathscr{A}} q_i = 0$, we get

$$\sum_i b_i^{(n-1)} \otimes x_i.\lambda(q_i \otimes_{\mathscr{B}} p_j) = 0$$

for each j. By continuity we have

$$\sum_i b_i^{(n-1)} \otimes r_j \otimes x_i.a_{ij} = \sum_i b_i^{(n-1)} \otimes r_j \otimes x_i.\lambda(\zeta_i \otimes_{\mathscr{B}} p_j) = 0.$$

It follows that

$$\begin{aligned} d_h(u) &= \sum_i b_i^{(n-1)} \otimes \left(\sum_j a_{ij}.r_j\right) \otimes x_i \\ &= \sum_i b_i^{(n-1)} \otimes \left(\sum_j \lambda(\zeta_i \otimes_{\mathscr{B}} p_j).r_j\right) \otimes x_i \\ &= \sum_i b_i^{(n-1)} \otimes \left(\sum_j \zeta_i.\mu(p_j \otimes_{\mathscr{A}} r_j)\right) \otimes x_i \\ &= \sum_i b_i^{(n-1)} \otimes \zeta_i.b \otimes x_i \\ &= \sum_i b_i^{(n-1)} \otimes q_i \otimes x_i \end{aligned}$$

proving that the cycle bounds. □

Lemma 6.4. *$\mathscr{H}_n(\mathscr{A}, \mathscr{A} \hat{\otimes} X) = (0)$ for any $X \in$ **mod**-$\mathscr{A}$ and $n \geq 1$.*

Proof. We shall use a similar application of Cohen's factorization theorem to define a 'local contracting homotopy'. First note that if we for $\alpha \in \mathscr{A}$ define $s_\alpha^n : C_n(\mathscr{A}, \mathscr{A} \hat{\otimes} X) \longrightarrow C_{n+1}(\mathscr{A}, \mathscr{A} \hat{\otimes} X)$ by

$$s_\alpha^n(a_1 \otimes \cdots \otimes a_n \otimes a \otimes x) = a_1 \otimes \cdots \otimes a_n \otimes a \otimes \alpha \otimes x$$

we have, denoting the Hochschild differentials by ∂_n,

$$\begin{aligned} &\partial_n s_\alpha^n(a_1 \otimes \cdots \otimes a_n \otimes a \otimes x) \\ &\quad = s_\alpha^{n-1}\partial_{n-1}(a_1 \otimes \cdots \otimes a_n \otimes a \otimes x) + (-1)^n a_1 \otimes \cdots \otimes a\alpha \otimes x \end{aligned}$$

If $\partial_{n-1}(\sum_i a_1^i \otimes \cdots \otimes a_n^i \otimes a^i \otimes x^i) = 0$, then, using Cohen's factorization theorem, we may write $a^i = b^i \alpha$ with $(b^i)_{i=1}^\infty \in ((a^i).\mathscr{A})^-$ and $\sum_i a_1^i \otimes \cdots \otimes a_n^i \otimes b^i \otimes x^i$ absolutely convergent. By continuity we have

$$\partial_{n-1}\left(\sum_{i=1}^{\infty} a_1^i \otimes \cdots \otimes a_n^i \otimes b^i \otimes x^i\right) = 0$$

and therefore

$$\begin{aligned} &\partial_n\left(s_\alpha^n\left(\sum_{i=1}^{\infty} a_1^i \otimes \cdots \otimes a_n^i \otimes b^i \otimes x^i\right)\right) \\ &\quad = (-1)^n \sum_{i=1}^{\infty} a_1 \otimes \cdots \otimes a_n^i \otimes b^i\alpha \otimes x^i \\ &\quad = (-1)^n \sum_{i=1}^{\infty} a_1^i \otimes \cdots \otimes a_n^i \otimes a^i \otimes x^i. \qquad \Box \end{aligned}$$

We return to the proof of acyclicity of the horizontal chain complexes for $n \geq 1$. By the first lemma we have exactness of

$$C_{2n} \xrightarrow{d_n} C_{1n} \xrightarrow{d_n} C_{0n} \longrightarrow 0$$

Using the Rieffel isomorphism $\mathscr{A} \hat{\otimes}_{\mathscr{A}} Q \cong {}_{\mathscr{A}}Q$, the complex

$$\cdots \longrightarrow C_{m+1,n} \longrightarrow C_{m,n} \longrightarrow \cdots \longrightarrow C_{2,n} \longrightarrow$$

is seen to be isomorphic to the complex

$$\begin{aligned} &\longrightarrow C_m(\mathscr{A}, \mathscr{A}_{\mathscr{A}} \hat{\otimes} \mathscr{B}^{\hat{\otimes}(n-1)} \otimes M) \hat{\otimes}_{\mathscr{A}} Q \\ &\longrightarrow C_{m-1}(\mathscr{A}, \mathscr{A}_{\mathscr{A}} \hat{\otimes} \mathscr{B}^{\hat{\otimes}(n-1)} \hat{\otimes} M) \hat{\otimes}_{\mathscr{A}} Q \\ &\longrightarrow \cdots \longrightarrow C_0(\mathscr{A}, \mathscr{A}_{\mathscr{A}} \hat{\otimes} \mathscr{B}^{\hat{\otimes}(n-1)} \otimes M) \hat{\otimes}_{\mathscr{A}} Q \end{aligned}$$

The lemma together with flatness of ${}_{\mathscr{A}}Q$ gives the horizontal acyclicity. Vertical acyclicity follows symmetrically, so Lemma 6.1 completes the proof. $\Box$

Corollary 6.5. *Let $\mathscr{A} \approx \mathscr{B}$ and suppose $\mathscr{A}$ is amenable. Then $\mathscr{B}$ is amenable.*

Proof. Since $\mathscr{A}$ has a bounded approximate identity, it is amenable if and only if $\mathscr{H}_0(\mathscr{A}, X)$ is Hausdorff and $\mathscr{H}_n(\mathscr{A}, X) = (0)$ for all $n \geq 1$ and all $X \in$ ess-$\mathscr{A}$-**mod**-$\mathscr{A}$. □

More generally we have the following corollary.

Corollary 6.6. *Let $\mathscr{A}$ and $\mathscr{B}$ be Morita equivalent, let F be the equivalence functor of the categories of essential bimodules, and let $X \in$ ess-$\mathscr{A}$-**mod**-$\mathscr{A}$. Then the bounded Hochschild cohomologies $\mathscr{H}^n(\mathscr{A}, X^*)$ and $\mathscr{H}^n(\mathscr{B}, (FX)^*)$ are isomorphic.*

Proof. The dual of the double complex giving isomorphism of Hochschild homologies $\mathscr{H}_*(\mathscr{A}, X)$ and $\mathscr{H}_*(\mathscr{B}, FX)$ is a double complex in the third quadrant, vertically and horizontally acyclic, except possibly on the axes and with the Hochschild cochain complexes for bounded cohomology on the axes. □

Corollary 6.7. *Let $\mathscr{A}$ and $\mathscr{B}$ be Morita equivalent. Then the bounded simplicial homologies $\mathscr{H}_*(\mathscr{A}, \mathscr{A})$ and $\mathscr{H}_*(\mathscr{B}, \mathscr{B})$ are isomorphic, and the bounded cyclic homologies, $\mathscr{H}C_*(\mathscr{A})$ and $\mathscr{H}C_*(\mathscr{B})$, are isomorphic.*

Proof. Let ${}_{\mathscr{B}}P_{\mathscr{A}}$ and ${}_{\mathscr{A}}Q_{\mathscr{B}}$ be the modules providing the equivalence. Then ${}_{\mathscr{B}}\mathscr{B}_{\mathscr{B}} \cong {}_{\mathscr{B}}P \hat{\otimes}_{\mathscr{A}} \mathscr{A} \hat{\otimes}_{\mathscr{A}} Q_{\mathscr{B}}$, so that the first statement follows from Theorem 6.2. Since $\mathscr{A}$ and $\mathscr{B}$ have bounded approximate identities, the long exact sequence of Connes exists [35, 12], yielding the validity of the second assertion. □

Remark. The version of bounded Hochschild homology for non-unital algebras we treat, is what in [18] is called 'naive Hochschild homology'. The main reason for this is that naive bounded Hochschild homology is predual to bounded Hochschild cohomology with coefficients in dual modules, the latter being of long standing importance in the theory of Banach algebras. If Hochschild homology of a non-unital algebra is defined by extension of functors, then the long exact sequence of Connes is automatically transferred from the unitization and thus holds for all (Banach) algebras, see [19, 35]. We are grateful to M. Wodzicki for having brought this point to our attention.

7. Some applications

We would like to apply the theory to algebras of compact operators. Let $\mathscr{F}(X)$ be the algebra of operators on a Banach space X uniformly approximable by finite rank operators, let $\mathscr{K}(X)$ be the Banach algebra of compact operators, and let $\mathscr{B}(X)$ be the Banach algebra of all bounded operators. In order for $\mathscr{F}(X)$ to have a bounded approximate identity it is sufficient and necessary that X^* has the bounded approximation property, and similarly $\mathscr{K}(X)$ has a bounded approximate identity if and only if X^* has the bounded $\mathscr{K}(X)^a$-approximation property (that is, the identity on X^* can be approximated in the topology of uniform convergence on compacta by adjoints of operators in $\mathscr{K}(X)$), see [11, 31].

First we note an immediate consequence of Corollary 4.3 and Corollary 6.5. In [11] it is shown that if $\mathscr{K}(X)$ and $\mathscr{K}(X^*)$ have bounded approximate identities, then $\mathscr{K}(X)$ and $\mathscr{K}(X^*)$ are simultaneously amenable. The proof uses the fact that, if $\mathscr{K}(X)^{\mathrm{op}}$ is represented as the left ideal of adjoint operators in $\mathscr{K}(X^*)$, we have $\mathscr{K}(X)^{\mathrm{op}}\mathscr{K}(X^*) = \mathscr{K}(X^*)$. Exploiting this fact we actually have the following corollary.

Corollary 7.1. *Suppose that $\mathscr{K}(X)$ and $\mathscr{K}(X^*)$ have bounded approximate identities. Then $\mathscr{K}(X)^{\mathrm{op}} \approx \mathscr{K}(X^*)$. In particular $\mathscr{K}(X)$ is amenable if and only if $\mathscr{K}(X^*)$ is amenable.*

As an other application we look at [10, Theorem 6.2]. Slightly reformulated it says in the context of algebras of the kind $\mathscr{F}(X)$: Let X and Y be two Banach spaces. Suppose that $\mathscr{F}(X)$ and $\mathscr{F}(Y)$ both have bounded approximate identities and that the product maps $\mathscr{F}(X,Y)\hat{\otimes}\mathscr{F}(Y,X)\longrightarrow\mathscr{F}(Y)$ and $\mathscr{F}(Y,X)\hat{\otimes}\mathscr{F}(X,Y)\longrightarrow\mathscr{F}(X)$ both are surjective. Then $\mathscr{F}(X)$ is amenable if and only if $\mathscr{F}(Y)$ is amenable.

In order to shed more light on this we start by finding the irreducible $\mathscr{F}(X)$-modules. We need a definition.

Definition 7.2. Let X be an arbitrary Banach space. Let F be a closed subspace of X^* and let I be a closed left ideal of $\mathscr{F}(X)$. Define

$$\Phi(F) = \operatorname{clspan}\{x \otimes \xi^* \mid x \in X,\ \xi^* \in F\}$$

where $x \otimes \xi^*$ is the notation for rank-1 operators. Define also

$$\Psi(I) = \{\xi^* \in X^* \mid \exists x \in X\backslash(0)\colon x \otimes \xi^* \in I\}.$$

Proposition 7.3. *Suppose X has the approximation property. Then Φ is a lattice isomorphism between the lattice of closed subspaces of X^* and the lattice of closed left ideals in $\mathscr{F}(X)$. The lattice inverse is Ψ.*

Proof. Clearly $\Phi(F)$ is a closed left ideal of $\mathscr{F}(X)$, so Φ is a lattice homomorphism between the two lattices we consider. Since I is a left ideal and X is irreducible over $\mathscr{F}(X)$ we may as well describe $\Psi(I)$ by $\Psi(I) = \{\xi^* \in X^* \mid \forall x \in X\colon x \otimes \xi^* \in I\}$. It is then immediate that $\Psi(I)$ is a closed subspace of X^*, so that Ψ is a lattice homomorphism between the two lattices.

Let I be an arbitrary closed left ideal in $\mathscr{F}(X)$. Then clearly $\Phi(\Psi(I)) \subseteq I$. Since X has the approximation property every operator in I is the limit of a sequence of finite rank operators from I. It follows that I is the closed span of its rank-1 operators, so we cannot have strict inclusion: $x \otimes \xi^* \in I \Rightarrow \xi^* \in \Psi(I) \Rightarrow x \otimes \xi^* \in \Phi(\Psi(I))$.

Let now F be a closed subspace of X^*. Then

$$\xi^* \in F \Leftrightarrow x \otimes \xi^* \in \Psi(F) \quad \forall x \in X.$$

The implication $\Rightarrow$ follows from the definition of Φ. Hence suppose $x \neq 0$ and $x \otimes \xi^* \in \Psi(F)$. Then there is a sequence of finite rank operators $S_n = \sum_i x_i^n \otimes f_i^n$ with all f_i^n's in F such that $S_n \longrightarrow x \otimes \xi^*$. Choose $x^* \in X^*$ so that $\langle x, x^* \rangle = 1$ and put $\xi_n^* = \sum_i \langle x_i^n, x^* \rangle f_i^n$. Then $\xi_n^* \in F$ and $x \otimes x^* S_n = x \otimes \xi_n^*$, so $\xi_n^* \longrightarrow \xi^*$. Since F is closed we have $\xi^* \in F$. Since, by definition of Ψ,

$$x \otimes \xi^* \in \Phi(F)\ \forall x \in X \Leftrightarrow \xi^* \in \Psi\phi(F)$$

we have $F = \Psi(\Phi(F))$. □

We can now prove that X is the only irreducible module over $\mathscr{F}(X)$.

Proposition 7.4. *Let E be an irreducible module over $\mathscr{F}(X)$ and suppose that X has the approximation property. Then $E \cong X$ as a module.*

Proof. Let L be a maximal modular left ideal of $\mathscr{F}(X)$ such that $E \cong \mathscr{F}(X)/L$. Since $L \neq \mathscr{F}(X)$ and since Ψ is a lattice isomorphism we have $\Psi(L) \neq X^*$. Choose $\xi^* \in X^* \setminus \Psi(L)$ and let $Q: \mathscr{F}(X) \longrightarrow \mathscr{F}(X)/L$ be the canonical map. By Proposition 7.3, the map $x \longrightarrow Q(x \otimes \xi^*)$ is a bounded non-zero module map between the irreducible modules X and E. By Schur's lemma and Banach's isomorphism theorem it is an isomorphism of Banach-$\mathscr{F}(X)$-modules. □

Theorem 7.5. *Let X and Y be Banach spaces such that X^* and Y^* both have the bounded approximation property. Then $\mathscr{F}(X) \approx \mathscr{F}(Y)$ if and only if the canonical maps*

$$\mathscr{F}(X,Y) \hat{\otimes} \mathscr{F}(Y,X) \longrightarrow \mathscr{F}(Y),$$

$$\mathscr{F}(Y,X) \hat{\otimes} \mathscr{F}(X,Y) \longrightarrow \mathscr{F}(X),$$

are both surjective.

Proof. Let $\mathscr{A} = \mathscr{F}(X)$ and $\mathscr{B} = \mathscr{F}(Y)$, and assume that the maps are surjective. Then ${}_{\mathscr{B}}P_{\mathscr{A}} = \mathscr{F}(X,Y)$ and ${}_{\mathscr{A}}Q_{\mathscr{B}} = \mathscr{F}(Y,X)$ are equivalence modules. Suppose conversely that there is a full Morita context $(\mathscr{A}, \mathscr{B}, Q, P, \mu, \lambda)$. Since an equivalence preserves irreducible modules (Proposition 5.1(3)) it follows from Proposition 7.4 that $P \hat{\otimes}_{\mathscr{A}} X \cong Y$ as modules over $\mathscr{B}$ and $Q \hat{\otimes}_{\mathscr{B}} Y \cong X$ as modules over $\mathscr{A}$. Making these identifications we see that each $p \in P$ represents an operator $X \longrightarrow Y$ by $x \longrightarrow p \otimes_{\mathscr{A}} x$ of norm not exceeding $\|p\|$. It follows that each operator $S \in \mathscr{F}(X)$ can be written $S = \sum T_i S_i$, $S_i \in B(X,Y)$, $T_i \in \mathscr{B}(Y,X)$ such that $\sum \|T_i\| \|S_i\| < \infty$. Since S can be written as a product of 3 operators from $\mathscr{F}(X)$ it follows that we may choose

$S_i \in \mathscr{F}(X,Y)$ and $T_i \in \mathscr{F}(Y,X)$. By symmetry we get the other half of the statement. □

We would like to use the algebras $\mathscr{F}(X)$ and modules $\mathscr{F}(X,Y)$ as a further illustration of the concept of a Morita context. First we determine the spaces of homomorphisms.

Proposition 7.6. *Let $\mathscr{A} = \mathscr{F}(X)$ and let Y,Z be two Banach spaces. Then $\mathbf{h}_{\mathscr{A}}(\mathscr{F}(X,Y),\mathscr{F}(X,Z))$ and $\mathscr{B}(Y,Z)$ are isometrically isomorphic.*

Proof. Let $S \in \mathscr{B}(Y,Z)$. Then $\varphi_S(T) = ST \quad (T \in \mathscr{F}(X,Y))$ defines a module homomorphism, and clearly $\|\varphi_S\| \le \|S\|$. Conversely, suppose that φ is a bounded module homomorphism, and choose $x_0 \in X$, $x_0^* \in X^*$ such that $x_0^*(x_0) = 1$ and $\|x_0\| = \|x_0^*\| = 1$. Then $S_\varphi(y) = \varphi(y \otimes x_0^*)x_0$ defines an operator $S_\varphi \in \mathscr{B}(Y,Z)$ with $\|S_\varphi\| \le \|\varphi\|$. Now $S_{\varphi_S}(y) = \varphi_S(y \otimes x_0^*)(x_0) = S(y)x_0^*(x_0) = S(y)$. We also have $\varphi_{S_\varphi}(y \otimes x^*) = (S_\varphi y) \otimes x^* = (\varphi(y \otimes x_0^*)x_0) \otimes x^* = \varphi((y \otimes x_0^*)(x_0 \otimes x^*)) = \varphi(y \otimes x^*)$. Since $\operatorname{clspan}\{y \otimes x^*\} = \mathscr{F}(X,Y)$ it follows that $\varphi_{S_\varphi} = \varphi$. □

Proposition 7.7. *Let $\mathscr{A} = \mathscr{F}(X)$ and let $P_{\mathscr{A}} = \mathscr{F}(X,Y)$. Suppose that $\mathscr{A}$ has a bounded approximate identity. Then $P_{\mathscr{A}}$ is a generator if and only if $\mathscr{F}(Y,X)\hat{\otimes}\mathscr{F}(X,Y) \longrightarrow \mathscr{F}(X)$ is surjective, and $P_{\mathscr{A}}$ is an approximate progenerator if and only if $\mathscr{F}(X,Y)\hat{\otimes}\mathscr{F}(Y,X) \longrightarrow \mathscr{F}(Y)$ is also surjective and Y has the bounded approximation property. If $P_{\mathscr{A}}$ is an approximate progenerator, then the Banach algebra $P\hat{\otimes}_{\mathscr{A}}P^{\#}$ is isomorphic to $\mathscr{F}(Y)$.*

Proof. Since ${}_{\mathscr{A}}P^{\#} = \overline{\mathscr{F}(X).\mathscr{B}(Y,X)} = \mathscr{F}(Y,X)$ by the previous proposition, the first statement follows from Proposition 4.7.

Suppose now that $P_{\mathscr{A}}$ is an approximate progenerator. According to Theorem 4.11 this implies that $\mathscr{F}(X,Y)\hat{\otimes}_{\mathscr{A}}\mathscr{F}(Y,X)$ is isomorphic to a closed two-sided ideal of $\mathscr{B}^{\#} = \mathscr{B}(Y)$ with a bounded left approximate identity. Since this ideal obviously consists of approximable operators, $\mathscr{F}(X,Y)\hat{\otimes}_{\mathscr{A}}\mathscr{B}(Y,X) \cong \mathscr{F}(Y)$. It follows that $\mathscr{F}(Y)$ has a bounded left approximate identity, so that Y has the bounded approximation property. This proves the 'only if'. Conversely, if Y satisfies the hypothesis, then $\mathscr{F}(X,Y)\hat{\otimes}_{\mathscr{A}}\mathscr{F}(Y,X) \cong \mathscr{F}(Y)$, so that Theorem 4.11 tells us that $P_{\mathscr{A}}$ is an approximate progenerator. □

Example 7.8. Let $\mathscr{C}_p$ $(p = 0, 1 \le p < \infty)$ be the spaces introduced by Johnson [16]. In [17] it is shown that $\mathscr{C}_0$ has a shrinking basis and that $\mathscr{C}_0^{**} = \mathscr{C}_1^*$ does not have the approximation property. The space C_p has for each $p = 0, 1 \le p < \infty$ the property that every approximable operator factors through it. First note that for any Banach space the product map $\mathscr{F}(X^{**}, X) \hat{\otimes} \mathscr{B}(X, X^{**}) \longrightarrow \mathscr{F}(X)$ is surjective, [8, VI.4.2]. Put $X = \mathscr{C}_0$ and $Y = \mathscr{C}_0^{**}$. Then $P_{\mathscr{A}} = \mathscr{F}(X, Y)$ is a generator for $\mathscr{A} = \mathscr{F}(X)$, $P \hat{\otimes}_{\mathscr{A}} P^{\#} = \mathscr{F}(Y)$, but $\mathscr{F}(Y)$ does not have a bounded left approximate identity. Hence both conditions on Y in Proposition 7.7 are necessary.

Now let X and $\mathscr{A}$ be as before, but put $Y = \mathscr{C}_1 = X^*$. Then $\mathscr{B} = P \hat{\otimes}_{\mathscr{A}} P^{\#} = \mathscr{F}(Y)$, has a bounded left approximate identity, but not a bounded right approximate identity [11, 31]. The opposite algebra $\mathscr{A}^{\mathrm{op}}$ is, by taking adjoint maps, isomorphic to a left ideal of $\mathscr{B} = \mathscr{F}(Y)$ so that $\mathscr{A}^{\mathrm{op}}\mathscr{B} = \mathscr{B}$. By Corollary 4.12, $\mathscr{A} \approx \mathscr{A}^{\mathrm{op}}$. The algebra $\mathscr{F}(X)$ is not Morita equivalent to $\mathscr{F}(Y)$, simply because the latter does not have a bounded approximate identity. Hence the existence of bounded approximate identities has to be stated explicitly in Theorem 7.5.

8. Concluding remarks

One of the reasons that Morita equivalence is a powerful tool is its usefulness in computing invariants. This usefulness of course rests on the fact that the Morita equivalence classes in general are much larger than the isomorphism classes. It is therefore an important task to establish a Morita theory for as large a class of Banach algebras as possible, while still keeping important Morita invariants. From this point of view it is perhaps disconserting that the algebra $\mathbb{C}$ of complex numbers is only Morita equivalent to the matrix algebras $M_n(\mathbb{C})$. Example 4.15 suggests that $\mathbb{C}$ ought to be Morita equivalent to algebras of nuclear operators on Banach spaces with the approximation property, providing an analogue of Morita equivalence between $\mathbb{C}$ and $\mathscr{K}(\mathscr{H})$ in the C^*-algebra context. The property of $\mathbb{C}$ of being homologically separable is of course then no longer Morita invariant, but there will be other advantages. Most notably perhaps, a theory including algebras of nuclear operators has good prospects of providing a setting to discuss induced representations and abstract imprimitivity theorems along the lines of [25]. As an example, let us look at the question of which bounded (non-involutive) representations of a discrete group G are induced from the trivial subgroup by means of $\ell_1(G)$ viewed as a bimodule in $\ell_1(G)$-**mod**-$\mathbb{C}$. If it is possible to regard $\ell_1(G)_{\mathbb{C}}$ as an equivalence module we should be able to determine the bounded G-modules induced from $\mathbb{C}$ by means of Morita equivalence between $\mathbb{C}$ nd $\mathscr{N}(\ell_1(G))$, the nuclear operators on the Banach space $\ell_1(G)$. We intend to return to this in a subsequent paper.

We have in this paper laid the framework for a general theory. It is our hope that it will serve as a means for classification for general Banach algebras as well as for specific classes. To illustrate this, a possible classifying element could be 'finite-infinite'. Thus, the content of [10, Theorem 6.8] can be interpreted in this way. Reformulated in the spirit of the present paper it says that if $\mathscr{A}$ is an amenable Banach algebra with trivial virtual centre and $p, q \in \mathscr{M}(\mathscr{A})$ are two complementary projections in the multiplier algebra, then $\mathscr{A}$ is Morita equivalent to either $p\mathscr{A}p$ or $q\mathscr{A}q$.

This property is of course also shared by algebras $\mathscr{F}(X)$ with bounded approximate identities and X approximately primary as discussed before Theorem 6.9 of [10]. Considering this as an analogue of Murray–von Neumann equivalence of projections in von Neumann algebras, these algebras should be termed 'infinite'. Esben Kehlet has kindly pointed out to us that, if $\mathscr{R}$ is a von Neumann algebra other than $\mathbb{C}$, then $\mathscr{R}$ is an infinite factor if and only if it has the property that for every projection $p \in \mathscr{R}$ either p or $1 - p$ is equivalent to 1. In particular infinite factors are infinite in above sense.

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